EC 2'41 W 9af INDEXED

+

THE QUARTERLY JOURNAL OF MATHEMATICS

(OXFORD SERIES)

Volume 12 *No.* 46 *June* 1941

OEA

CONTENTS

OXFORD

AT THE CLARENDON PRESS

1941

Price 7s. 6d. net

PRINTED IN GREAT BRITAIN BY JOHN JOHNSON AT THE OXFORD UNIVERSITY PRESS

THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SERIES

Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, J. H. C. THOMPSON

With the co-operation of A. L. DIXON, W. L. FERRAR, G. H. HARDY E. A. MILNE, E. C. TITCHMARSH

THE QUARTERLY JOURNAL OF MATHEMATICS (OXFORD SERIES) is published in March, June, September, and December, at a price of 7s. 6d. net for a single number with an annual subscription (for four numbers) of 27s. 6d. post free.

Papers, of a length normally not exceeding 20 printed pages of the Journal, are invited on subjects of Pure and Applied Mathematics, and should be addressed 'The Editors, Quarterly Journal of Mathematics, Clarendon Press, Oxford'. Contributions can be accepted in French and German, if in typescript (formulae excepted). Authors of papers printed in the Quarterly Journal will be entitled to 50 free offprints. Correspondence on the *subject-matter* of the Quarterly Journal should be addressed, as above, to 'The Editors', at the Clarendon Press. All other correspondence should be addressed to the Publisher (Humphrey Milford, Oxford University Press, at the temporary address).

HUMPHREY MILFORD

OXFORD UNIVERSITY PRESS

AMEN HOUSE, LONDON, E.C.4

Temporary address:

SOUTHFIELD HOUSE, HILL TOP ROAD, OXFORD

INVERSION IN GROUPS

By OLGA TAUSSKY (*London*) and J. TODD (*London*)

[Received 29 January 1941]

We shall be concerned with $(1, 1)$ transformations of a group G into itself. The class of all these forms a group when they are combined in the usual manner, by superposition; thus

$$UV = U\{V\}.$$

Inversion with respect to $t \in G$ is the transformation

$$x \to I_t(x) = tx^{-1}t;$$

inversion with respect to the unit element in G, which is the transformation $x \to x^{-1}$, is denoted by I. Right-hand and left-hand translations with respect to $t \in G$ are the transformations

$$x \to T_r(x) = xt, \qquad x \to T_l(x) = tx.$$

We observe that $$I_t = T_r I T_r^{-1}. \tag{1}$$*

The two sections below are quite independent.

1. B. de Kerekjarto† has obtained the following result, which has uses in topology.

Theorem. *Let A be an Abelian group which contains no elements of order 2. Then inversion with respect to the unit element in G is characterized by the following properties:*

(*a*) *the unit element is the only invariant element;*

(*b*) *the transformation obtained by superposing the conjugate, with respect to any translation, of the inversion on the inversion itself is a certain translation.*

In virtue of (1) this theorem enables a characterization of inversion with respect to a general element of G to be given. We shall establish the following result:

Theorem.‡ *Let G be any§ group. Then inversion with respect to the unit element in G is characterized by the following properties:*

(*c*) *the unit element is an invariant element*;

* There is no ambiguity in the notation T_r^{-1} since $(I)_r^{-1} = (T^{-1})_r$.

† *Comptes Rendus* (Paris), 210 (1940), 288–9.

‡ As before, using (1), we can give a characterization of inversion with respect to an arbitrary element in G.

§ G need not be Abelian and it may contain elements of order 2: the elements of order 2, if any, are invariant, as well as the unit element.

(*d*) *the transformation obtained by superposing the conjugate of the inversion with respect to any right-hand translation* T_r *on the inversion itself can also be obtained by applying a right-hand and a left-hand translation, both with respect to the element t in question.*

Proof. Let F be a transformation satisfying (*c*) and (*d*). Then, from (*d*), when t is the unit element in G, it is obvious that F^2 is the identical transformation. From (*d*), for an arbitrary t, we have

$$T_r F T_r^{-1} F = T_r T_l, \tag{2}*$$

from which, using the fact that $F = F^{-1}$, we find that

$$T_r F = T_r T_l F T_r. \tag{3}$$

Operating with both sides of (3) on the unit element in G we obtain

$$t^{-1} = F(t),$$

which establishes the theorem.

2. In this section we shall determine the class of groups in which distance, in the sense of Menger,† is invariant under inversion.

Let G be any group. Then the distance between two elements a, b of G is the class of two elements (ab^{-1}, ba^{-1}): this distance is therefore symmetric and invariant under right-hand translation. Suppose now that distances in G are invariant under inversion with respect to $t \in G$. The condition that the distance between a and b is invariant is that

$$(ab^{-1}, ba^{-1}) = (ta^{-1}bt^{-1}, tb^{-1}at^{-1}),$$

and this implies either

$$ab^{-1} = ta^{-1}bt^{-1} \tag{4}$$

or

$$ab^{-1} = tb^{-1}at^{-1}. \tag{5}$$

The condition that the distance between the unit element and ab^{-1} should be invariant is that

$$(ab^{-1}, ba^{-1}) = (tba^{-1}t^{-1}, tab^{-1}t^{-1}),$$

and this implies either

$$ab^{-1} = tba^{-1}t^{-1} \tag{6}$$

or

$$ab^{-1} = tab^{-1}t^{-1}. \tag{7}$$

From (4) and (6) or from (5) and (7) it follows that $ab = ba$, while from (5) and (6) or from (4) and (7) it follows that $a^2 = b^2$.

* The first member of (2) is the commutator of T_r and F since F^2 is the identical transformation.

† *Math. Zeitschrift*, **33** (1931), 396–418.

We have therefore shown that any two elements a, b of G either commute or satisfy $a^2 = b^2$. If a, b do not commute, neither do a and b^{-1}, nor do a and ab. Hence

$$a^2 = b^2 = b^{-2} = (ab)^2,$$

and this shows that a, b generate a quaternion group. Hence it follows that G is a Hamiltonian 2-group.*

Since it is obvious that distances in any Hamiltonian 2-group are invariant under inversion with respect to any element, we have established the following theorem:†

THEOREM. *A condition necessary and sufficient for a group to be Abelian or a Hamiltonian 2-group is that distances in it should be invariant under inversion with respect to any element.*

* A *2-group* is a group in which the order of each element is a power of 2.

† This is a generalization of a previous result [O. Taussky, *Math. Annalen*, 108 (1933), 615–20].

INTEGRATION OF THE DIFFERENTIAL EQUATIONS OF APPELL'S FUNCTION F_4

By A. ERDÉLYI (*Edinburgh*)

[Received 12 February 1941; in revised form 30 March 1941]

1. In the following pages I deal with the integration of the system of partial differential equations of which Appell's fourth hypergeometric function of two variables F_4 is a solution. F_4 has the remarkable property of reducing to the product of two ordinary hypergeometric functions when the parameters are connected by the relation $\alpha+\beta+1 = \gamma+\gamma'$.* More recently Burchnall (**4**) discussed this exceptional case from a different point of view. He introduced new variables suggested by the Watson-Bailey transformation and showed that the introduction of these new variables splits up the system of partial differential equations, in the exceptional case, into two ordinary differential equations. Using this result, Burchnall discussed the exceptional case fully.

Studying Burchnall's paper, I came to the conclusion that his variables should have some significance in the general case of the system of F_4 as well as in the reducible case. Transforming the (general) equations of F_4 to the new variables, I had to realize, however, that the integration of the new system is still a difficult problem. Later, in connexion with some research on the integration of systems of partial linear differential equations, I found that the differential equations of F_4 can be integrated by contour integrals. I propose to outline this method in the present paper. The technique of the integration of differential equations by definite integrals being sufficiently well known, I shall omit the details.

First I find a new integral representation of F_4. This integral representation suggests the integral which can be shown to represent the general solution of the system of F_4. This integral in its turn suggests the introduction of new variables which prove to be identical with those introduced by Burchnall. Also I transform my integral into a double integral. Since the first version of this paper was written, Burchnall and Chaundy have published (**5**) some most interesting investigations into double hypergeometric series. In particular, Burchnall and Chaundy gave a double-integral representation of F_4

* Watson (**7**), Bailey (**3**).

which is more symmetrical than mine. Though the two double integrals resemble each other, there does not seem to be a simple transformation of the variables of integration connecting the two representations. I conclude by sketching the fundamental systems of solutions of the system of F_4 and its monodromic group.

I mention only in passing that the methods used here apply to corresponding hypergeometric functions of higher orders as well as to those of more than two variables. For instance, the system of n partial linear differential equations of order $q+1$ in the n independent variables $x_1,\ldots, x_n$,

$$\frac{\partial}{\partial x_k}\left(\gamma_k-1+x_k\frac{\partial}{\partial x_k}\right)(\beta_1+\theta)\ldots(\beta_{q-1}+\theta)z-$$
$$-(\alpha_1+\theta)\ldots(\alpha_p+\theta)z = 0 \quad (k = 1, 2,\ldots, n),$$

where $\theta = x_1\dfrac{\partial}{\partial x_1}+\ldots+x_n\dfrac{\partial}{\partial x_n}$, one solution of which is

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(\alpha_1)_{m_1+\ldots+m_n}\cdots(\alpha_p)_{m_1+\ldots+m_n}x_1^{m_1}\ldots x_n^{m_n}}{m_1!\ldots m_n!(\beta_1)_{m_1+\ldots+m_n}\cdots(\beta_{q-1})_{m_1+\ldots+m_n}(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}$$

may be integrated by the $(n-1)$-fold integral

$$\int\cdots\int dt_1\ldots dt_{n-1}\,t_1^{-\gamma_1}\ldots t_{n-1}^{-\gamma_{n-1}}(1-t_1-\ldots-t_{n-1})^{-\gamma_n}\times$$
$$\times{}_pF_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_{q-1},\gamma_1+\ldots+\gamma_n-n+1;\right.$$
$$\left.\frac{x_1}{t_1}+\ldots+\frac{x_{n-1}}{t_{n-1}}+\frac{x_n}{1-t_1-\ldots-t_{n-1}}\right).$$

Analogous results for hypergeometric functions of slightly different type readily follow.

2. Appell's series (**1**), (**2**)

$$F_4(\alpha,\beta;\gamma,\gamma';x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{m+n}}{m!\,n!\,(\gamma)_m(\gamma')_n}x^my^n \tag{1}$$

satisfies the system of two partial differential equations*

$$L_1[z] \equiv x(1-x)r-y^2t-2xys+\{\gamma-(\alpha+\beta+1)x\}p-$$
$$-(\alpha+\beta+1)yq-\alpha\beta z = 0,$$
$$L_2[z] \equiv y(1-y)t-x^2r-2xys+\{\gamma'-(\alpha+\beta+1)y\}q-$$
$$-(\alpha+\beta+1)xp-\alpha\beta z = 0, \tag{2}$$

* (**2**), 44.

where p, q, r, s, t are Monge's notation for

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$$

respectively.

The integral representation mentioned in the preceding section reads

$$F_4(\alpha,\beta;\gamma,\gamma';x,y) = \Gamma(\gamma)\Gamma(\gamma')\Gamma(2-\gamma-\gamma')e^{(\gamma+\gamma')\pi i}(2\pi i)^{-2}\times$$
$$\times \int\limits_P t^{-\gamma}(1-t)^{-\gamma'}F\left(\alpha,\beta;\gamma+\gamma'-1;\frac{x}{t}+\frac{y}{1-t}\right)dt, \quad (3)$$

where P denotes a Pochhammer double-loop slung round the points 0 and 1 such that $|x/t+y/(1-t)| < 1$ along the contour.* This integral representation is easily proved by expanding the hypergeometric function in the integrand and integrating term by term.

Now (3) suggests that

$$z = \int\limits_C t^{-\gamma}(1-t)^{-\gamma'}f\left(\frac{x}{t}+\frac{y}{1-t}\right)dt \quad (4)$$

should be a solution of (2) when C is any closed contour and $f(\omega)$ is any branch of Riemann's function defined by the scheme

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & \omega \\ 2-\gamma-\gamma' & \gamma+\gamma'-\alpha-\beta-1 & \beta & \end{matrix}\right\}. \quad (5)$$

Also it is to be expected that (4) should represent the general solution of (2), i.e. that any solution of (2) should be represented by linear combinations of integrals of the type (4).

To prove the first part of this conjecture let us write

$$x/t+y/(1-t) = \omega.$$

Now, from (4),

$$L_1[z] = \int\limits_C t^{-\gamma}(1-t)^{-\gamma'}\left[\left\{\frac{x(1-x)}{t^2}-\frac{y^2}{(1-t)^2}-\frac{2xy}{t(1-t)}\right\}f''+\right.$$
$$\left.+\left\{\frac{\gamma}{t}-(\alpha+\beta+1)\frac{x}{t}-(\alpha+\beta+1)\frac{y}{1-t}\right\}f'-\alpha\beta f\right]dt$$
$$= \int\limits_C t^{-\gamma}(1-t)^{-\gamma'}[\omega(1-\omega)f''+\{\gamma+\gamma'-1-(\alpha+\beta+1)\omega\}f'-\alpha\beta f]\,dt -$$
$$-\int\limits_C d\{t^{-\gamma}(1-t)^{1-\gamma'}f'\},$$

* See (8), § 12.43.

by integration by parts. The first integral vanishes in consequence of the differential equation* satisfied by (5). Hence

$$L_1[z] = -\int_C d\{t^{-\gamma}(1-t)^{1-\gamma'}f'\}, \tag{6}$$

and similarly

$$L_2[z] = -\int_C d\{t^{1-\gamma}(1-t)^{-\gamma'}f'\}. \tag{7}$$

From this result it is seen that (4) is certainly a solution of (2) whenever C is either a closed contour or else an open contour at the two ends of which both $t^{-\gamma}(1-t)^{1-\gamma'}f'(\omega)$ and $t^{1-\gamma}(1-t)^{-\gamma'}f'(\omega)$ vanish.

The second part of the above conjecture can be proved by showing that four linearly independent solutions of (2)—e.g. the four given on page 52 of (**2**)—may be represented by integrals of the type (4). I omit the details of this proof.

3. It is now a straightforward matter to derive the complete theory of the solutions of (2) from (4). In fact the well-known method of integrating linear differential equations by contour integrals† combined with the equally well-known theory of Riemann's P-function‡ furnishes us with the fundamental systems of solutions, transformations connecting different fundamental systems, and also with the monodromic group of the system (2). I shall mention only a few points of interest here.

The possible singularities of the integrand of (4) are given by the singularities of the P-function, i.e. by $\omega = \infty$, 0, or 1. There are six corresponding values of t, viz. 0, 1, $x/(x-y)$, ∞, and the two roots of the quadratic equation in t:

$$t(1-t)-x(1-t)-yt = 0. \tag{8}$$

The last two singularities are the only ones which are *not* rational functions of x and y. Obviously it is possible to express all six singularities as rational functions of some new variables, for instance the roots of (8), which will be denoted by X and Y.

From (8) we have

$$x = XY, \qquad y = (1-X)(1-Y), \tag{9}$$

and hence the new variables, introduced in order to 'rationalize' the singularities of (5), are identical with the variables used by Burchnall (**4**) in the reducible case $\alpha+\beta+1 = \gamma+\gamma'$. The six possible singu-

* (**8**), § 10.7. † See e.g. (**6**), chap. 18. ‡ See e.g. (**8**), chap. 14.

larities of the integrand of (4) are now $t = 0, 1, X, Y, XY/(X+Y-1)$, ∞. Also

$$\omega = \frac{x}{t}+\frac{y}{1-t} = 1-\frac{(t-X)(t-Y)}{t(t-1)}$$

and hence, from a well-known transformation formula of Riemann's P-function,*

$$f(\omega) = P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & 1-\dfrac{(t-X)(t-Y)}{t(t-1)} \\ 2-\gamma-\gamma' & \gamma+\gamma'-\alpha-\beta-1 & \beta & \end{matrix}\right\}$$

$$= P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & \dfrac{(t-X)(t-Y)}{t(t-1)} \\ \gamma+\gamma'-\alpha-\beta-1 & 2-\gamma-\gamma' & \beta & \end{matrix}\right\}. \qquad (10)$$

Either (5) or (10) may be used in connexion with (4).

4. Now z as given by (4) is a regular function of x and y (or of X and Y) unless two singularities of the integrand which should be separated by C coincide. Hence all possible singularities of (4), i.e. all singular curves of (2), are given by the coincidence of two of the six singularities enumerated above. Thus we arrive at the list of singular curves

$$X = 0, 1, \infty; \qquad Y = 0, 1, \infty; \qquad X = Y. \qquad (11)$$

At first it may seem that there is another singular curve,

$$X+Y-1 = 0,$$

this equation being the condition for the coincidence of the two singularities $t = \infty$ and $t = XY/(X+Y-1)$ of the integrand. However, $X+Y-1 = 0$ is *not* a singularity. For, according to (10),

$$f(\omega) = aF(\alpha,\beta;\gamma+\gamma'-1;\omega)+$$
$$+b\omega^{2-\gamma-\gamma'}F(\alpha-\gamma-\gamma'+2,\beta-\gamma-\gamma'+2;3-\gamma-\gamma';\omega),$$

and correspondingly $z = az_1+bz_2$, where a and b are constants. It is easily seen that the integrand of z_1 is regular at $t = XY/(X+Y-1)$ and the integrand of z_2 at $t = \infty$; hence the coincidence of these two points cannot possibly cause any singular behaviour of z.

On the other hand, according to the current theory, the singular

* (8), § 10.72. Here (10) means that any particular branch of the P-function in the first line is equal to some branch of the P-function in the second line.

curves of the system (2) can be found by solving (2) for r and t, thus getting

$$r = \frac{2xys}{x(1-x-y)}+..., \qquad t = \frac{2xys}{y(1-x-y)}+...,$$

where the dots signify terms containing no derivatives of the second order. The possible singular curves of (2) are (i) the singularities of the coefficients, i.e. $x = 0$, $y = 0$, $x = \infty$, $y = \infty$, $x+y = 1$, and (ii) the curves determined by $1-4xy(1-x-y)^{-2} = 0$. Thus the usual theory* yields, besides (11), also $x+y = 1$, which from our result is seen to be merely an apparent singular curve.

5. The P-function (10) is represented by an integral of the form†

$$f(\omega) = \int s^{\gamma+\gamma'-\beta-2}(s-1)^{1-\gamma-\gamma'+\alpha}\left\{s-\frac{(t-X)(t-Y)}{t(t-1)}\right\}^{-\alpha} ds.$$

Hence, instead of (4), we may represent the solutions of (2) by the double integral

$$z = \iint s^{\gamma+\gamma'-\beta-2}(s-1)^{1-\gamma-\gamma'+\alpha}t^{\alpha-\gamma}(t-1)^{\alpha-\gamma'}\times$$
$$\times\{st(t-1)-(t-X)(t-Y)\}^{-\alpha}\, dsdt, \qquad (12)$$

which represents solutions of (2) whenever the contours of integration in the s- and t-planes are suitably chosen: this follows from the way in which we obtained it. Closed contours in particular are always suitable contours.

Introducing new variables of integration σ, τ by the substitution

$$s = \frac{\sigma\tau-(1-\sigma+\tau)X}{\sigma(\sigma-1)}, \qquad t = \frac{\sigma\tau}{1-\sigma+\tau},$$

(12) changes into

$$z = \iint \sigma^{\beta-\gamma}(\sigma-1)^{\beta-\gamma'}(\sigma-X)^{1-\gamma-\gamma'+\alpha}\tau^{\alpha-\gamma}(\tau+1)^{\alpha-\gamma'}(\tau+Y)^{-\alpha}\times$$
$$\times\{(\sigma-X)\tau-X(1-\sigma)\}^{\gamma+\gamma'-\alpha-\beta-1}\, d\sigma d\tau. \qquad (13)$$

This form of the solution of (2), as well as the more symmetrical double-integral representation due to Burchnall and Chaundy,‡ shows directly that in the exceptional case $\gamma+\gamma'-\alpha-\beta-1 = 0$ the typical solutions of (2) are products of hypergeometric functions of X and Y and, moreover, that for positive integral values of $\gamma+\gamma'-\alpha-\beta-1$ the typical solutions of (2) are finite sums of products of ordinary hypergeometric functions.

* Cf. e.g. (**2**), § xii. † (**8**), § 14.6. ‡ (**5**), equation (68).

6. Although I do not propose to work out the fundamental systems of solutions of (2) in detail, a few remarks on these solutions may not be without interest. Incidentally, these remarks illustrate a point in the general analytic theory of systems of partial differential equations which does not seem to have been noted before.

The singular points of (2) fall into two classes:

(i) the points $x = y = 0$ (i.e. $X = Y-1 = 0$ or $X-1 = Y = 0$), $x = y^{-1} = 0$ (i.e. $X = Y^{-1} = 0$ or $Y = X^{-1} = 0$), and $y = x^{-1} = 0$ (i.e. $X-1 = Y^{-1} = 0$ or $Y-1 = X^{-1} = 0$) which are *ordinary intersections* of *two singular curves* of the regular type;

(ii) the points $x = 1-y = 0$, $1-x = y = 0$, and $x = y = \infty$ which are *points of contact* of the two singular curves of the regular type passing through them. Therefore these singular points are of a far more complicated type. In fact, in order to deal with these singular points, we have to introduce the new variables X, Y, thus getting the singular points $X = Y = 0$, $X = Y = 1$, and $X = Y = \infty$, each of which is an *ordinary intersection* of *three singular curves*.

The fundamental systems of solutions belonging to the singular points (i) are easily determined. Indeed, the fundamental system for $x = y = 0$ is known.* *One* solution for the vicinity of $y = x^{-1} = 0$ is given by (4) when we choose

$$f(\omega) = \omega^{-\alpha} F(\alpha, \alpha-\gamma-\gamma'+2; \alpha+1-\beta; \omega^{-1})$$

and take C to be a Pochhammer double-loop encircling 0 and 1, such that along it $|\omega| > 1$. This solution turns out to be

$$x^{-\alpha} F_4(\alpha, \alpha-\gamma+1; \alpha-\beta+1, \gamma'; 1/x, y/x).$$

Now, this result suggests that by linear transformations the singular points $x = y^{-1} = 0$ and $y = x^{-1} = 0$ can be transformed into $x = y = 0$. Hence the above-cited result yields the fundamental systems belonging to each of the singular points of class (i). The transformations which these systems undergo when the variables encircle the corresponding singularity are easily inferred from the analytical expression of the solutions. The transformations connecting two different fundamental systems follow from a known transformation-formula.†

7. The singular points (ii) are of a more involved type. There is no fundamental system of solutions convergent in the *whole neigh-*

* (2), 52.

† (2), 26, equation (37).

bourhood of such a singular point. Introducing X, Y, we see that all three singularities (ii) are equivalent, i.e. transform into each other by the transformations mentioned in the last paragraph. Therefore it is sufficient to deal with $X = Y = 0$.

There are two linearly independent solutions of (2) convergent in the entire neighbourhood of $X = Y = 0$. We obtain them by taking in (4)

$$f(\omega) = F(\alpha, \beta; \gamma+\gamma'-1; \omega), \tag{14}$$

C being the loop $\infty...(1+)...\infty$, and

$$f(\omega) = \omega^{2-\gamma-\gamma'}F(\alpha-\gamma-\gamma'+2, \beta-\gamma-\gamma'+2; 3-\gamma-\gamma'; \omega),$$

C being a double-loop encircling 0 and $XY/(X+Y-1)$. The first of these solutions is a power-series in X, Y, the second is $(XY)^{1-\gamma}$ multiplied by a power-series in X, Y. They do not seem to be hypergeometric series of two variables.

In order to complete the fundamental system we have to resolve, as it were, the hyperspherical neighbourhood of $X = Y = 0$ into three hyperconical neighbourhoods, i.e. we have to take different fundamental systems according as $|X|$, $|Y|$, or $|X-Y|$ is the smallest modulus. 'Near $X = 0$', i.e. when $|X|$ is the smallest of these three moduli, we shall have a further solution taking (14) with C to be a double-loop encircling 0 and X. This solution is of the form $X^{1-\gamma}$ multiplied by a one-valued function of X, Y (in the domain of sufficiently small values of $|X| < |Y|$). Again, the fourth solution is given by

$$f(\omega) = (1-\omega)^{\gamma+\gamma'-\alpha-\beta-1}\times$$
$$\times F(\gamma+\gamma'-\alpha-1, \gamma+\gamma'-\beta-1; \gamma+\gamma'-\alpha-\beta; 1-\omega), \tag{15}$$

and C is a loop beginning and ending at $t = Y$ and encircling the points 0, X, $XY/(X+Y-1)$. This solution is of the form $Y^{1-\gamma}$ multiplied by a one-valued function of X, Y. The connexion between this fundamental system and that discussed by Burchnall (**4**), in the exceptional case $\gamma+\gamma' = \alpha+\beta+1$, is obvious.

'Near $Y = 0$', i.e. when $|Y| < |X|$ and both are small, and also 'near $X = Y$', i.e. when, say, $|X-Y| < |X|$ and both are small, we have to choose the two solutions belonging to the pairs of indices $(1-\gamma, 0)$ and $(0, 1-\gamma)$ in a corresponding way. The characteristic feature of these fundamental sets is that they form a canonical set only in a hypercone with vertex $X = Y = 0$, and not in the entire

hyperspherical neighbourhood of that singular point. This feature arises because more than two singular curves intersect in that point.

8. I conclude with a few remarks on the monodromic group of the system (2). The singular curves in the finite part of the (X,Y)-space are the five curves $X=0$, $X=1$, $Y=0$, $Y=1$, $X=Y$.

When (X,Y) encircles one of the two singular curves $X=0$, $Y=0$ (and no other singular curve) in the positive direction, the 'first fundamental system' referred to in § 6* undergoes a canonical substitution characterized by the 4×4 diagonal matrix S_1 with diagonal elements 1, $e^{-2\pi i\gamma}$, 1, $e^{-2\pi i\gamma}$. Similarly, when (X,Y) encircles $X=1$ or $Y=1$, the same fundamental system undergoes a substitution characterized by the diagonal matrix S_2 with diagonal elements 1, 1, $e^{-2\pi i\gamma'}$, $e^{-2\pi i\gamma'}$.

In order to deal with the singular curve $X=Y$, the introduction of a 'second fundamental system' is advisable. From our results it is easily inferred that there are *three* linearly independent solutions of (2) which are one-valued in the vicinity of $X-Y=0$. These are the solutions in which the two singularities $t=X$ and $t=Y$ of the integrand of (4) are both inside or both outside the same loop of the contour. Furthermore, there is a fourth solution, where $f(\omega)$ is given by (15) and C is a double-loop encircling $t=X$ and $t=Y$, which obviously is of the form $(Y-X)^{2(\gamma+\gamma'-\alpha-\beta)-1}$ multiplied by a function one-valued in the vicinity of $Y-X=0$. These four solutions constitute a fundamental system for $X=Y$. The corresponding canonical substitution is given by a diagonal matrix S_3 with diagonal elements 1, 1, 1, $e^{4\pi i(\gamma+\gamma'-\alpha-\beta-1)}$ respectively.

Denoting by A the 4×4 matrix which transforms the 'second' fundamental system into the 'first', it is easily seen that the monodromic group of (2) is generated by the three substitutions S_1, S_2, and AS_3A^{-1}.

9. We remark that (3) is by no means the only representation of F_4 by a simple integral. There are several others, e.g. one of the type

$$\int u^{\beta-1}(1-u)^{\gamma'-\beta-1}(1-uy)^{-\alpha}F\left[\alpha,\beta-\gamma'+1;\gamma;\frac{-ux}{(1-u)(1-ux)}\right]du$$

which could be used for integrating (2). However, (3) is the most symmetrical of the integral representations known to me.

* Cf. (2), 52.

REFERENCES

1. P. Appell, *Comptes Rendus*, Paris, 90 (1880), 296–8 and 731–4.
2. —— and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques: Polynomes d'Hermite* (Paris, 1926).
3. W. N. Bailey, *Quart. J. of Math.* (Oxford), 4 (1933), 305–8.
4. J. L. Burchnall, ibid. 10 (1939), 145–50.
5. —— and T. W. Chaundy, ibid. 11 (1940), 249–70.
6. E. L. Ince, *Ordinary Differential Equations* (London, 1927).
7. G. N. Watson, *Proc. London Math. Soc.* (2) 20 (1922), 189–95.
8. E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1927).

78

ON DIRICHLET'S SINGULAR INTEGRAL AND FOURIER TRANSFORMS

By H. KOBER (*Birmingham*)

[Received 3 March 1941]

1. LET $f(t) \in L_p$ over $(-\infty, \infty)$, let

$$|f|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p\,dt\right)^{1/p} \quad \text{or} \quad |f|_p = \operatorname*{ess.u.b.}_{-\infty<t<\infty} |f(t)|$$

for $0 < p < \infty$ or $p = \infty$ respectively, and, for $1 \leqslant p \leqslant 2$, let $\phi(x) = Ff$ denote the Fourier transform of $f(t)$*

$$\phi(x) = \underset{a\to\infty}{\overset{\text{index } p'}{\text{l.i.m.}}}\, (2\pi)^{-\frac{1}{2}} \int_{-a}^{a} f(t)e^{-itx}\,dt \quad \left(\frac{1}{p}+\frac{1}{p'}=1\right). \tag{1.1}$$

E. Hille and J. D. Tamarkin have proved† that, for $1 < p \leqslant 2$, the following inversion formula corresponds to (1.1):

$$f(t) = \underset{a\to\infty}{\overset{\text{index } p}{\text{l.i.m.}}}\, (2\pi)^{-\frac{1}{2}} \int_{-a}^{a} \phi(x)e^{itx}\,dx. \tag{1.2}$$

The tool that they employ is Dirichlet's singular integral

$$D_\alpha f = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha(t-s)}{t-s} f(t)\,dt \quad \begin{pmatrix} 0<\alpha<\infty, \\ -\infty<s<\infty \end{pmatrix}. \tag{1.3}$$

This operator has been dealt with by various writers.‡ Hille and Tamarkin show that, for $1 < p < \infty$, $D_\alpha f$ has the properties

$$|D_\alpha f|_p \leqslant A_p |f(t)|_p, \qquad |D_\alpha f - f(s)|_p \to 0 \text{ as } \alpha \to \infty. \tag{1.4}$$

* E. C. Titchmarsh, *Proc. London Math. Soc.* (2) 23 (1923), 279–89. When $p = 1$, obviously the right-hand side of (1.1) is equivalent to the ordinary integral over $(-\infty, \infty)$.

† E. Hille and J. D. Tamarkin, *Bull. American Math. Soc.* 39 (1933), 768–74. For $p = 2$, the result is due to M. Plancherel.

‡ e.g. H. Bateman, *Proc. London Math. Soc.* (2) 4 (1906), 461–98; G. H. Hardy, *Proc. London Math. Soc.* (2) 7 (1909), 445–72; G. H. Hardy and E. C. Titchmarsh, *Proc. London Math. Soc.* (2) 23 (1924), 1–26, and 30 (1929), 95–106; E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 1.14 and 11.21; E. Hille, *Trans. American Math. Soc.* 39 (1936), 131–53, and *Proc. National Ac. of Sci.* 24 (1938), 159–61, and *Annals of Math.* 40 (1939), 1–47. Theorems 1 (*c*) and 5 (*a*) of my former paper on Dirichlet's operator, *Quart. J. of Math.* (Oxford), 11 (1940), 66–80, can be deduced from some results due to Hardy, loc. cit., §§ 11 and 16.

When $p = 1$, neither (1.4) nor, therefore, (1.2) holds; for, guided by the analogy with Fourier series, Hille and Tamarkin have constructed an example which shows that $f(t) \in L_1$ does not imply $D_\alpha f \in L_1$. Hence the following problems arise: If $f(t) \in L_1$,

(*a*) under what conditions on $f(t)$ does $D_\alpha f$ belong to L_1:

(*b*) under what conditions is the Fourier transformation, applied to $f(t)$, inversible in the mean-convergence sense, i.e. in the sense of (1.2), with $p = 1$?

It turns out that it is only under rather restrictive conditions on $f(t)$ that $D_\alpha f$ belongs to L_1 (§§ 2–7, Theorems 1 and 2); Theorem 1 (ii) is due to H. R. Pitt. The answer to problem (*b*) is given by Theorem 3; though it is fairly simple, it appears not to have been remarked up to now.

2. Theorem 1. (i) *Let $f(t) \in L_1(-\infty, \infty)$ and $0 < \alpha < \infty$. A necessary condition that $D_\alpha f$ belongs to $L_1(-\infty, \infty)$ is that*

$$\int_{-\infty}^{\infty} f(t)e^{-it\alpha}\,dt = \int_{-\infty}^{\infty} f(t)e^{it\alpha}\,dt = 0. \tag{2.1}$$

(ii) *The condition is not sufficient.*

Proof. Let $\phi(x) = Ff$. Then $\phi(x)$ is continuous in $(-\infty, \infty)$, and*

$$f(s, \alpha) = D_\alpha f = (2\pi)^{-\frac{1}{2}} \int_{-\alpha}^{\alpha} \phi(x)e^{isx}\,dx. \tag{2.2}$$

Suppose also that $f(s, \alpha) = D_\alpha f$ belongs to $L_1(-\infty, \infty)$. Then the function $\psi(x) = F[f(t, \alpha)]$ is continuous in $(-\infty, \infty)$ as well. By the uniqueness theorem we have almost everywhere

$$\psi(x) = \phi(x) \quad (-\alpha < x < \alpha), \qquad \psi(x) = 0 \quad (|x| > \alpha). \tag{2.3}$$

By continuity the equations (2.3) also hold in the strict sense. Since $\psi(x)$ is continuous at $x = \pm\alpha$, we have $\phi(\alpha) = \phi(-\alpha) = 0$, which proves Theorem 1 (i).

Let $f(t) \in L_1(-\infty, \infty)$ and $\phi(\alpha) = \phi(-\alpha) = 0$. H. R. Pitt shows in a paper to be published shortly that the conditions $g(a) = g(b) = 0$, where $g(x) = Fh$, $h(t) \in L_1$, and $-\infty < a < b < \infty$, do not imply the existence of a function $h_1(t) \in L_1$ such that its Fourier transform $\psi(x)$ is equal to $g(x)$ in (a, b) and to zero outside (a, b). This result is equivalent to Theorem 1 (ii).

* Cf. H. Kober, loc. cit. 73.

3. Let $f(t)$ be a step function which satisfies (2.1). Computing $D_\alpha f$, we can easily see that it belongs to $L_p(-\infty,\infty)$ for any $p > \frac{1}{2}$ and that, when r, ρ are any numbers such that $r(2-\rho)$, $0 \leqslant \rho < 2$, $s^\rho D_\alpha f$ belongs to $L_r(-\infty,\infty)$. In the general case we obtain the following result:

THEOREM 2. *Let* (i) $f(t) \in L_1(-1,1)$, (ii) $t^\rho f(t) \in L_1(-\infty,\infty)$ *for some* $\rho > 0$, (iii) $f(t)$ *satisfy* (2.1). *Then* $D_\alpha f$ *belongs to* $L_1(-\infty,\infty)$; *more generally,* $s^{\rho'} D_\alpha f \in L_r(-\infty,\infty)$ *when* $0 \leqslant \rho' < \min(1,\rho)$ *and*

$$r\{1-\rho'+\min(\rho,1)\} > 1.$$

We need some lemmas.

LEMMA 1. *Let* $f(t) \in L_1(-\infty,\infty)$, *then* $D_\alpha f$ *belongs to* $L_r(-\infty,\infty)$ *for any* r *such that* $1 < r \leqslant \infty$.*

LEMMA 2. *Let* $0 < \rho < 1$. *Then, for* $-\infty < x,\, y < \infty$,

$$\{\Gamma(\rho)\}^{-1} \int_x^{\to\infty} (t-x)^{\rho-1} e^{-ity}\, dt = \exp\{-\tfrac{1}{2}i\pi\rho \operatorname{sgn} y - ixy\}|y|^{-\rho}, \tag{3.1}$$

$$\{\Gamma(\rho)\}^{-1} \int_{\to-\infty}^{x} (x-t)^{\rho-1} e^{ity}\, dt = \exp\{-\tfrac{1}{2}i\pi\rho \operatorname{sgn} y + ixy\}|y|^{-\rho}. \tag{3.2}$$

LEMMA 3. *Let* $0 < \rho \leqslant 1$, *let* $f(t)$ *and* $t^\rho f(t)$ *belong to* $L_1(-\infty,\infty)$, *and let* $\phi(x) = Ff$; *let*

$$\phi_\rho(x) = F[t^\rho f(t)] = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} t^\rho f(t) e^{-itx}\, dt, \tag{3.3}$$

where $\arg t = -\pi$ *for* $t < 0$, *that is to say,* $t^\rho = e^{-i\pi\rho}|t|^\rho$ *for* $t < 0$. *Then*

$$\phi(x) = \{\Gamma(\rho)\}^{-1} e^{i\pi\rho/2} \int_x^{\to\infty} (\xi - x)^{\rho-1} \phi_\rho(\xi)\, d\xi. \tag{3.4}$$

We remark that, according to Weyl's definition,† $\exp(\frac{1}{2}i\pi\rho)\phi_\rho(x)$ is the derivative of $\phi(x)$ of order ρ, and that the lemma does not hold for any $\rho > 1$.

4. We can easily prove Lemma 2 by the substitutions $t = x \pm t'$. The proof of Lemma 3 for $\rho = 1$ is an easy consequence of the Riemann-Lebesgue theorem; therefore we suppose that $0 < \rho < 1$.

* H. Kober, loc. cit., Theorem 1 (*a*).

† *Vierteljahrschr. d. Naturf. Ges. Zürich*, 62 (1917), 296–302.

By absolute convergence, we have, for $M \geqslant x$,

$$I_\rho^{(M)} = \frac{1}{\Gamma(\rho)} \int_x^M (\xi - x)^{\rho-1} \phi_\rho(\xi)\, d\xi$$

$$= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} y^\rho f(y) \frac{dy}{\Gamma(\rho)} \int_x^M (\xi - x)^{\rho-1} e^{-i\xi y}\, d\xi. \quad (4.1)$$

We can now show that the function

$$Z(x, y, M) = |y|^\rho \int_x^M (\xi - x)^{\rho-1} e^{-i\xi y}\, d\xi \quad (4.2)$$

is bounded uniformly in x, y, and M; (3.4) will then follow from (3.1) and (4.1) by the Lebesgue convergence theorem. For $y > 0$ and $M > x+y^{-1}$, we have

$$Z(x, y, M) = \int_{xy}^{My} (\xi - xy)^{\rho-1} e^{-i\xi}\, d\xi = \int_{xy}^{xy+1} + \int_{xy+1}^{My} = I_1 + I_2, \quad (4.3)$$

and $|I_1| \leqslant \rho^{-1}$, while $|I_2| \leqslant 4$ by the second mean-value theorem; hence Z is bounded. The case $x \leqslant M \leqslant x+y^{-1}$ is trivial. The case $y < 0$ follows in a similar way. Thus we have proved the lemma.

5. Proof of Theorem 2

Without loss of generality we may suppose $0 < \rho < 1$; for, if Theorem 2 is true for $0 < \rho < 1$, then it is obviously true for any $\rho \geqslant 1$. By (2.2) and by Lemma 3,

$$f(s, \alpha) = \frac{(2\pi)^{-\frac{1}{2}} e^{i\pi\rho/2}}{\Gamma(\rho)} \int_{-\alpha}^{\alpha} e^{isx}\, dx \int_x^{\to\infty} (\xi - x)^{\rho-1} \phi_\rho(\xi)\, d\xi \quad (-\infty < s < \infty).$$

Now, for $M \geqslant x$, the function

$$W(x, M) = \int_x^M (\xi - x)^{\rho-1} \phi_\rho(\xi)\, d\xi = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t) t^\rho\, dt \int_x^M (\xi - x)^{\rho-1} e^{-it\xi}\, d\xi$$

is bounded uniformly, in consequence of (4.2), (4.3). Therefore, by an argument similar to that of § 4, we have

$$f(s, \alpha) = \frac{(2\pi)^{-\frac{1}{2}} e^{i\pi\rho/2}}{\Gamma(\rho)} \int_{-\alpha}^{\to\infty} \phi_\rho(\xi)\, d\xi \int_{-\alpha}^{\min(\xi, \alpha)} (\xi - x)^{\rho-1} e^{isx}\, dx,$$

and so

$$f(s,\alpha) = \frac{(2\pi)^{-\frac{1}{2}}e^{i\pi\rho/2}}{\Gamma(\rho)}\Biggl\{\int\limits_{-\alpha}^{\alpha} \phi_\rho(\xi)\,d\xi \int\limits_{\to-\infty}^{\xi} (\xi-x)^{\rho-1}e^{isx}\,dx -$$

$$-\int\limits_{-\alpha}^{\alpha} \phi_\rho(\xi)\,d\xi \int\limits_{\to-\infty}^{-\alpha} (\xi-x)^{\rho-1}e^{isx}\,dx +$$

$$+\int\limits_{\alpha}^{\to\infty} \phi_\rho(\xi)\,d\xi \int\limits_{-\alpha}^{\alpha} (\xi-x)^{\rho-1}e^{isx}\,dx\Biggr\} = I_1+I_2+I_3.$$

By (3.2) and (2.2),

$$I_1 = \frac{e^{i\pi\rho(1-\operatorname{sgn} s)/2}}{(2\pi)^{\frac{1}{2}}|s|^\rho}\int\limits_{-\alpha}^{\alpha} \phi_\rho(\xi)e^{is\xi}\,d\xi = \frac{e^{i\pi\rho(1-\operatorname{sgn} s)/2}}{|s|^\rho} D_\alpha\{t^\rho f(t)\}.$$

Since $t^\rho f(t)$ belongs to $L_1(-\infty,\infty)$, $D_\alpha\{t^\rho f(t)\}$ belongs to $L_r(-\infty,\infty)$ for any $r > 1$ (Lemma 1). Taking $r = 2(2-\rho+\rho')^{-1}$, where $0 \leqslant \rho' < \rho$, we arrive at

$$s^{\rho'}I_1 \in L_1(-\infty,-1), \qquad s^{\rho'}I_1 \in L_1(1,\infty)$$

with the aid of Hölder's theorem. We have still to show that

$$s^{\rho'}(I_2+I_3) \in L_1(-\infty,-1), \qquad s^{\rho'}(I_2+I_3) \in L_1(1,\infty).$$

Integrating by parts, we have

$$I_2 = -\frac{(2\pi)^{-\frac{1}{2}}e^{i\pi\rho/2}}{is\Gamma(\rho)}\int\limits_{-\alpha}^{\alpha} \phi_\rho(\xi)\,d\xi\Biggl\{e^{-is\alpha}(\xi+\alpha)^{\rho-1} -$$

$$-(1-\rho)\int\limits_{-\infty}^{-\alpha}(\xi-x)^{\rho-2}e^{isx}\,dx\Biggr\} = I_{21}+I_{22}.$$

$$I_3 = \frac{(2\pi)^{-\frac{1}{2}}e^{i\pi\rho/2}}{is\Gamma(\rho)}\int\limits_{\alpha}^{\to\infty} \phi_\rho(\xi)\,d\xi\Biggl\{e^{is\alpha}(\xi-\alpha)^{\rho-1}-e^{-is\alpha}(\xi+\alpha)^{\rho-1} -$$

$$-(1-\rho)\int\limits_{-\alpha}^{\alpha} e^{isx}(\xi-x)^{\rho-2}\,dx\Biggr\} = I_{31}+I_{32}+I_{33}.$$

By Lemma 3, we have

$$I_{31} = (2\pi)^{-\frac{1}{2}}(is)^{-1}e^{is\alpha}\phi(\alpha); \quad I_{21}+I_{32} = -(2\pi)^{-\frac{1}{2}}(is)^{-1}e^{-is\alpha}\phi(-\alpha),$$

and so, by (2.1), $I_{31} = 0$, $I_{21}+I_{32} = 0$.

Thus it only remains to prove that both $s^{\rho'}I_{22}$ and $s^{\rho'}I_{33}$ belong to $L_1(-\infty,-1)$ and to $L_1(1,\infty)$. For it will then follow that $s^{\rho'}f(s,\alpha)$ belongs to $L_1(-\infty,\infty)$, since $f(s,\alpha) = D_\alpha f$ is bounded in $(-\infty,\infty)$.

The proof of the more general assertion of Theorem 2 does not present any further difficulties.

6. We now require

LEMMA 4. *Let* $0 < \xi \leqslant U < \infty$ *and* $0 < \rho < 1$, *let* s *be real, and let*

$$W(s,\xi,U) = \int_{\xi}^{U} v^{\rho-2}e^{-isv}\,dv.$$

Then, for $A = \max\{3,(1-\rho)^{-1}\}$ *and for any* σ *such that* $\rho-1 \leqslant \sigma \leqslant \rho$,

$$|W(s,\xi,U)| \leqslant A\xi^{\sigma-1}|s|^{\sigma-\rho}. \tag{6.1}$$

Proof. By direct estimation and by integration by parts, we have

$$|W| \leqslant (1-\rho)^{-1}\xi^{\rho-1}, \qquad |W| \leqslant 3\xi^{\rho-2}|s|^{-1},$$

and so

$$|W| = |W|^{(1-\rho+\sigma)+(\rho-\sigma)} \leqslant (A\xi^{\rho-1})^{1-\rho+\sigma}(A\xi^{\rho-2}|s|^{-1})^{\rho-\sigma} = A\xi^{\sigma-1}|s|^{\sigma-\rho},$$

which proves the lemma.

7. Investigation of I_{22} and I_{33}

Since $\phi_\rho(\xi)$ is bounded, we have

$$|I_{22}| < \frac{K}{|s|}\int_{-\alpha}^{\alpha} d\xi\left|\int_{-\infty}^{-\alpha}(\xi-x)^{\rho-2}e^{isx}\,dx\right| = \frac{K}{|s|}\int_{-\alpha}^{\alpha} d\xi\left|\int_{\xi+\alpha}^{\infty} v^{\rho-2}e^{-isv}\,dv\right|.$$

Hence
$$|I_{22}| < \frac{K}{|s|^{1+\rho'+\delta}}\int_{-\alpha}^{\alpha}(\xi+\alpha)^{\sigma-1}\,d\xi < \frac{K}{|s|^{1+\rho'+\delta}}$$

when, in Lemma 4, we take $\sigma = \rho-\rho'-\delta$, where $\delta = \frac{1}{2}(\rho-\rho')$. Similarly,

$$|I_{33}| < \frac{K}{|s|}\int_{\alpha}^{\infty} d\xi\left|\int_{\xi-\alpha}^{\xi+\alpha} v^{\rho-2}e^{-isv}\,dv\right| = \frac{K}{|s|}\left\{\int_{\alpha}^{\alpha+1} + \int_{\alpha+1}^{\infty}\right\} = I_4+I_5,$$

$$|I_4| < \frac{K}{|s|^{1+\rho'+\delta}}\int_{\alpha}^{\alpha+1}(\xi-\alpha)^{\sigma-1}\,d\xi < \frac{K}{|s|^{1+\rho'+\delta}} \quad \begin{pmatrix}\sigma = \rho-\rho'-\delta,\\ \delta = \frac{1}{2}(\rho-\rho')\end{pmatrix},$$

$$|I_5| < \frac{K}{|s|}\int_{\alpha+1}^{\infty}(\xi-\alpha)^{\rho-2}|s|^{-1}\,d\xi < \frac{K}{|s|^2} \quad (\sigma = \rho-1).$$

Hence both $s^{\rho'}I_{22}$ and $s^{\rho'}I_{33}$ belong to $L_q(-\infty,-1)$ and $L_q(1,\infty)$ for any q such that $1 \leqslant q \leqslant \infty$. Thus we have proved the theorem.

Finally we state

THEOREM 1′. *Let n be any positive integer, and let $f(t) \in L_1(-\infty, \infty)$. Then $D_\alpha^{(n)} f = d^n(D_\alpha f)/ds^n$ belongs to $L_1(-\infty, \infty)$ if and only if $D_\alpha f$ belongs to $L_1(-\infty, \infty)$.*

8. Inversibility of the Fourier transformation

THEOREM 3. *Let $f(t) \in L_1(-\infty, \infty)$. Then a necessary and sufficient condition that the transform*

$$\phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t) e^{-itx}\, dt \tag{8.1}$$

satisfies the inversion formula (1.2) with $p = 1$ is that, for $-\infty < t < \infty$, $f(t)$ is equivalent to an integral function of exponential type; if this condition is satisfied, then $f(t)$ is the Fourier transform of $\phi(-x)$ in the ordinary sense.

The equation (1.2) means that there is a number $c > 0$ such that the integral

$$I_\alpha = \int_{-\infty}^{\infty} \left| f(t) - (2\pi)^{-\frac{1}{2}} \int_{-\alpha}^{\alpha} \phi(x) e^{itx}\, dx \right| dt \tag{8.2}$$

exists for any $\alpha > c$, and that $I_\alpha \to 0$ as $\alpha \to \infty$.

The mere existence of the integral I_α requires that the function

$$f_\alpha(t) = (2\pi)^{-\frac{1}{2}} \int_{-\alpha}^{\alpha} \phi(x) e^{itx}\, dt \tag{8.3}$$

should belong to $L_1(-\infty, \infty)$. Now, by (8.1) and (2.2), the right-hand side of (8.3) is equal to $D_\alpha f$. Therefore, by Theorem 1 (i),

$$\phi(\alpha) = \phi(-\alpha) = 0.$$

Hence the mere existence of I_α for any $\alpha > c$ implies that $\phi(x)$ vanishes for $|x| > c$. Therefore $\phi(x) \in L_2(-\infty, \infty)$; in consequence of the Plancherel theory of Fourier transforms, we have

$$f(t) = (2\pi)^{-\frac{1}{2}} \int_{-c}^{c} \phi(x) e^{itx}\, dx. \tag{8.4}$$

When we replace t by $z = x+iy$, then obviously the right-hand side of (8.4) is an integral function of z, and its modulus is smaller than $K \exp(c|z|)$.

Conversely, let $f(z)$ be an integral function and $|f(z)| < Ke^{c|z|}$ and $f(t) \in L_1$. These hypotheses imply* that $\phi(x) = Ff$ vanishes for $|x| > c$. Therefore, we arrive again at (8.4). Hence $I_\alpha = 0$ for $\alpha \geqslant c$. This proves the theorem.

* See M. Plancherel and G. Pólya, *Comm. Math. Helvetici*, 9 (1936–7), 224–48, § 6. The corresponding result for $f(t) \in L_2$ is due to Paley and Wiener, *American Math. Soc. Coll. Pub.* 19 (1934), 12–13. Obviously there is complete reciprocity between f and ϕ if and only if $f \equiv \phi \equiv 0$.

NOTE ON CONTINUOUS INDEPENDENT FUNCTIONS

By A. C. OFFORD (*Bangor*)

[Received 17 March 1941]

THE *independence** of two real functions $f(x)$ and $g(x)$ is defined in the following way. We consider all sets of the form

$$E_1 = \mathcal{E}_x\{\alpha \leqslant f(x) \leqslant \beta\},$$
$$E_2 = \mathcal{E}_x\{\alpha' \leqslant g(x) \leqslant \beta'\},$$

where α, β and α', β' are any numbers and the signs $\leqslant$ may be replaced by $<$ in any of the inequalities. The functions $f(x)$ and $g(x)$ are said to be *independent* if

$$|E_1 E_2| = |E_1|\,|E_2|$$

for all possible sets E_1 and E_2.

It is known that there exist continuous functions which are independent.† On the other hand, it is well known that two continuous functions of bounded variation cannot be independent.

The object of this note is to find some condition which added to the condition of continuity makes independence impossible. Here there is no question of dropping the hypothesis of continuity, for, if one of the functions is permitted to have a discontinuity, then the construction of a pair of independent functions becomes trivial. Thus

$$f(x) = \begin{cases} 1 & (0 \leqslant x < \tfrac{1}{2}), \\ 0 & (\tfrac{1}{2} \leqslant x \leqslant 1), \end{cases} \qquad g(x) = \begin{cases} 2x & (0 \leqslant x \leqslant \tfrac{1}{2}), \\ 2-2x & (\tfrac{1}{2} \leqslant x \leqslant 1) \end{cases}$$

are independent although both are of bounded variation.

The condition we employ is akin to the condition (T_2) of Banach.‡ A finite function $g(x)$ is said to fulfil the condition (T_2) of Banach on an interval I if almost every one of its values is assumed at most an enumerable infinity of times on I. It is known that a function of bounded variation fulfils condition (T_2)§ and that any continuous function which fulfils the condition (N) of Lusin also fulfils condition (T_2).‖

* M. Kac, *Stud. Math.* 6 (1936), 46–58 (47). A similar definition had previously been given by Kolmogoroff, *Comptes Rendus de l'Acad. Communiste* (1929), 8–21. Cf. also Steinhaus, *Actualités Scientifiques*, 736 (1938), 57–73 (59).

† H. Steinhaus, *Comm. Math. Helvetici*, 9 (1936–7), 166–9.

‡ S. Saks, *Theory of the Integral*, 2nd ed. (Warsaw, 1937), 277.

§ Loc. cit. 279.

‖ Loc. cit. 284.

Our condition is not so stringent as (T_2). We say that a finite function $g(x)$ 'fulfils condition (T_2') on an interval I' if a tleast one of its values is assumed at most an enumerable infinity of times on I.

THEOREM. *Suppose that $f(x)$ and $g(x)$ are both continuous in the closed interval I and that neither is a constant. Then, if $g(x)$ fulfils condition (T_2') on I, the functions cannot be independent.*

COROLLARY. *If two continuous functions are independent in a closed interval I and if neither of them is a constant, then each function takes every one of its values an unenumerable infinity of times on I.*

Proof. Without loss of generality we may suppose that the closed interval I is the interval $0 \leqslant x \leqslant 1$. We denote by $f[E]$ and $g[E]$ the sets of values assumed by $f(x)$ and $g(x)$ respectively when x belongs to the set E.

Suppose that a is a value of $g(x)$ which $g(x)$ assumes at most an enumerable infinity of times in $(0, 1)$. Let H be the set of values of x for which $g(x) = a$; then, since $g(x)$ is continuous, H is a closed enumerable set.

Consider the set $f[H]$. This set is also enumerable and, since $f(x)$ is continuous and not a constant, there must be a value of $f(x)$, say b, which does not belong to the set $f[H]$. Consider the function

$$\phi(x) = |f(x)-b|$$

for x belonging to H. This function is continuous on H, and H is closed, and so $\phi(x)$ must assume its lower bound on H. But this lower bound cannot be zero, for then b would belong to $f[H]$. Hence there is a positive number δ such that

$$|f(x)-b| \geqslant 2\delta > 0$$

for all x of H. Consequently, if $b-\delta \leqslant y \leqslant b+\delta$ and $x \in H$,

$$|f(x)-y| \geqslant \delta.$$

Now write $$E_1 = \mathcal{E}_x\{b-\delta \leqslant f(x) \leqslant b+\delta\}, \tag{1}$$

then, in view of the fact that $f(x)$ is continuous, E_1 is closed and $|E_1| > 0$. Indeed, if x_0 is a value for which $f(x) = b$, then

$$|f(x)-b| \leqslant \delta \quad \text{for} \quad |x-x_0| \leqslant \eta$$

say, and so E_1 must contain the interval $(x_0-\eta, x_0+\eta)$. Further, the sets E_1 and H have no point in common.

Consider next the function $|g(x)-a|$ on the set E_1. This function is never zero on E_1 since E_1 contains no point of H. Also, since E_1 is closed and $g(x)$ continuous, it must attain its lower bound on E_1 so that

$$|g(x)-a| \geqslant 2\delta' > 0$$

for $x \in E_1$. Therefore, if

$$E_3 = \mathcal{E}_x\{a-\delta' < g(x) < a+\delta'\}, \tag{2}$$

E_3 is an open set of positive measure and has no point in common with E_1, so that E_1 is a part of the closed set CE_3.

Let us now apply the criterion for independence to the functions $f(x)$ and $g(x)$ taking for the interval (α, β) the interval $(b-\delta, b+\delta)$ and for the interval (α', β') the sum of the two intervals $g(x) \leqslant a-\delta'$ and $g(x) \geqslant a+\delta'$, which is clearly permissible. Then E_1 is the set defined by (1), and $E_2 = CE_3$ where E_3 is defined by (2).

Since E_1 and E_3 have no point in common, E_1 is part of E_2, and so

$$|E_1 E_2| = |E_1|.$$

On the other hand, we have proved that $|E_1| > 0$ and $|E_3| > 0$, and this latter implies $|E_2| < 1$. Hence

$$|E_1|\,|E_2| < |E_1|,$$

i.e.

$$|E_1 E_2| \neq |E_1|\,|E_2|,$$

and thus $f(x)$ and $g(x)$ are not independent.

The corollary is an immediate consequence.

ON EXPANSIONS IN EIGENFUNCTIONS (V)

By E. C. TITCHMARSH (*Oxford*)

[Received 30 March 1941]

1. In the previous paper* the case where a second-order differential equation leads to an expansion in a series of eigenfunctions was considered. We shall now consider the more general case in which the expansion takes the form of a Stieltjes integral.†

In the proof the Poisson-Stieltjes formula for a function analytic in a half-plane will be used in the following form:

Let $f(w)$ be an analytic function of $w = u+iv$, regular for $v > 0$. Let it be bounded on each line $v =$ constant, and let its maximum modulus on the line tend to 0 as $v \to \infty$. Let $f(w) = p(u,v)+iq(u,v)$, and

$$\int_{-\infty}^{\infty} |p(u,v)|\, du \leqslant M \quad (v > 0). \tag{1.1}$$

Then there is a function $\rho(x)$, of bounded variation in $(-\infty,\infty)$, such that

$$f(w) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\rho(x)}{x-w} \quad (v > 0). \tag{1.2}$$

Also

$$\lim_{v\to 0} \int_{u_1}^{u_2} p(u,v)\, du = \rho(u_2)-\rho(u_1) \tag{1.3}$$

for all values of u_1 and u_2.

The corresponding theorem for a circle is proved by Nevanlinna.‡ The case of the above theorem in which $p(u,v) \geqslant 0$ is proved by Lengyel.§ As I have not found the general case in the literature, I give a proof.

Integrating $f(z)/(z-w)$ along the straight line $(-R+iy, R+iy)$ and round the semicircle above it, where $0 < y < v$, and making $R \to \infty$, we obtain

$$f(w) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-R+iy}^{R+iy} \frac{f(z)}{z-w}\, dz. \tag{1.4}$$

* See above, pp. 33–50.

† H. Weyl, *Math. Annalen*, 68 (1910), 220–69.

‡ R. Nevanlinna, *Eindeutige analytische Funktionen*, 180–8.

§ B. Lengyel, 'On the spectral theorem of self-adjoint operators', *Acta Szeged*, 9 (1939), 174–86.

Similarly,
$$0 = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-R+iy}^{R+iy} \frac{f(z)}{z-w'}\,dz, \tag{1.5}$$
where $w' = u+i(2y-v)$. Subtracting the conjugate of (1.5) from (1.4), we get
$$f(w) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{p(x,y)}{x+iy-w}\,dx. \tag{1.6}$$

From this and (1.1) we deduce
$$|f(w)| \leqslant \frac{1}{\pi(v-y)} \int_{-\infty}^{\infty} |p(x,y)|\,dx \leqslant \frac{M}{\pi(v-y)},$$
and, making $y \to 0$,
$$|f(w)| \leqslant M/\pi v. \tag{1.7}$$
Now let
$$f_1(w) = \int_i^w f(z)\,dz = p_1(u,v)+iq_1(u,v),$$
$$f_2(w) = \int_i^w f_1(z)\,dz = p_2(u,v)+iq_2(u,v).$$

By (1.7), $f_1(w) = O\{\log(1/v)\}$ as $v \to 0$, uniformly over a finite u-interval. Hence $f_2(w)$ tends to a limit as $v \to 0$, uniformly over a finite u-interval. This limit $f_2(u) = p_2(u,0)+iq_2(u,0)$ is thus a continuous function of u.

Now
$$f_1(w) = -i\int_v^1 f(iy)\,dy + \int_0^u f(x+iv)\,dx,$$
and hence
$$p_1(u,v) = \int_v^1 q(0,y)\,dy + \int_0^u p(x,v)\,dx$$
$$= \int_v^1 q(0,y)\,dy + \chi(u,v) - \omega(u,v),$$
where
$$\chi(u,v) = \tfrac{1}{2}\int_0^u \{|p(x,v)|+p(x,v)\}\,dx,$$
$$\omega(u,v) = \tfrac{1}{2}\int_0^u \{|p(x,v)|-p(x,v)\}\,dx.$$
For each v, the functions χ and ω are non-decreasing functions of u, and $|\chi| \leqslant M$, $|\omega| \leqslant M$. Let
$$P(u,v,h) = \{p_2(u+h,v)-p_2(u,v)\}/h.$$

Then

$$P(u,v,h) = \frac{1}{h}\int\limits_u^{u+h} p_1(x,v)\,dx = \int\limits_v^1 q(0,y)\,dy + \chi_1(u,v,h) - \omega_1(u,v,h),$$

where

$$\chi_1(u,v,h) = \frac{1}{h}\int\limits_u^{u+h} \chi(x,v)\,dx, \qquad \omega_1(u,v,h) = \frac{1}{h}\int\limits_u^{u+h} \omega(x,v)\,dx.$$

For given u and v, the functions χ_1 and ω_1 are non-decreasing functions of h, and $|\chi_1| \leqslant M$, $|\omega_1| \leqslant M$. Hence, if $(h_\nu, h_\nu+\delta_\nu)$ are any non-overlapping intervals,

$$\sum_\nu |P(u,v,h_\nu+\delta_\nu) - P(u,v,h_\nu)| \leqslant \sum_\nu \{\chi_1(u,v,h_\nu+\delta_\nu) - \chi_1(u,v,h_\nu)\} +$$
$$+ \sum_\nu \{\omega_1(u,v,h_\nu+\delta_\nu) - \omega_1(u,v,h_\nu)\} \leqslant 4M.$$

Making $v \to 0$, it follows that

$$\sum_\nu |P(u,0,h_\nu+\delta_\nu) - P(u,0,h_\nu)| \leqslant 4M.$$

Hence $P(u,0,h)$ is of bounded variation, and so tends to a limit as $h \to \pm 0$. Thus $p_2(u,0)$ has everywhere right-hand and left-hand derivatives $p'_{2,+}(u,0)$ and $p'_{2,-}(u,0)$. Also

$$|P(u,v,h) - P(u,v,1)| \leqslant 4M.$$

Making $v \to 0$ we obtain

$$|P(u,0,h)| \leqslant 4M + |p_2(u+1,0)| + |p_2(u,0)|.$$

This is bounded in any finite interval. Hence $p_2(u,0)$ is absolutely continuous, and so is the integral of its derivative, which exists almost everywhere.

Further, χ_1 and ω_1 are non-decreasing functions of u, for given h and v. Hence, if $(u_\nu, u_\nu+\delta_\nu)$ are non-overlapping intervals,

$$\sum_\nu |P(u_\nu+\delta_\nu, v, h) - P(u_\nu, v, h)| \leqslant 4M.$$

Making $v \to 0$, then $h \to \pm 0$, it follows that $p'_{2,+}(u,0)$ and $p'_{2,-}(u,0)$ are of bounded variation in $(-\infty, \infty)$. Let

$$\rho(u) = \tfrac{1}{2}\{p'_{2,+}(u+0,0) + p_{2,+}(u-0,0)\}$$

(the same $\rho(u)$ would be obtained from $p'_{2,-}(u,0)$).

Integrating (1.6) by parts we have

$$f(w)=\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{p_1(x,y)}{(x+iy-w)^2}\,dx=\frac{2}{\pi i}\int\limits_{-\infty}^{\infty}\frac{p_2(x,y)}{(x+iy-w)^3}\,dx,$$

the integrated terms vanishing since $p_1(x,y)=O(1)$, $p_2(x,y)=O(x)$ for fixed y. In the last formula we can make $y\to 0$, and obtain

$$f(w)=\frac{2}{\pi i}\int\limits_{-\infty}^{\infty}\frac{p_2(x,0)}{(x-w)^3}\,dx. \tag{1.8}$$

To justify this step, we observe that

$$f_1(w)=\left(\int\limits_{i}^{u+i}+\int\limits_{u+i}^{u+iv}\right)f(z)\,dz=O(u)+O\left(\log\frac{1}{v}\right)$$

by (1.7). Hence

$$p_2(u,v)-p_2(u,0)=\mathbf{R}\int\limits_{0}^{v}if_1(u+iy)\,dy=O\left\{v\left(u+\log\frac{1}{v}\right)\right\}. \tag{1.9}$$

Hence, as $y\to 0$,

$$\int\limits_{-\infty}^{\infty}\frac{p_2(x,y)-p_2(x,0)}{(x+iy-w)^3}\,dx=O\int\limits_{-\infty}^{\infty}\frac{y(|x|+\log 1/y)}{|x|^3+v^3}\,dx=o(1);$$

and also

$$\int\limits_{-\infty}^{\infty}\left\{\frac{1}{(x+iy-w)^3}-\frac{1}{(x-w)^3}\right\}p_2(x,0)\,dx=o(1),$$

since $p_2(x,0)=O(x)$, by (1.9) with v fixed.

Integrating (1.8) by parts we have

$$f(w)=\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{p_2'(x,0)}{(x-w)^2}\,dx=\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{\rho(x)}{(x-w)^2}\,dx, \tag{1.10}$$

and (1.2) follows on integrating by parts again.

Since (1.10) is uniformly convergent, on integrating over (u_1,u_2) and taking real parts we have

$$\int\limits_{u_1}^{u_2}p(u,v)\,du=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left\{\frac{v}{(x-u_2)^2+v^2}-\frac{v}{(x-u_1)^2+v^2}\right\}\rho(x)\,dx.$$

(1.3) follows from this by the theory of Cauchy's singular integral.†

* See E. C. Titchmarsh, *Fourier Integrals*, 30–1.

2. Let $\psi(x)$, the arbitrary function to be expanded, be such that $\psi(x)$ and $L\psi(x)$ are $L^2(0,\infty)$, where

$$L \equiv \frac{d^2}{dx^2} - q(x);$$

and let
$$\psi(0)\cos h + \psi'(0)\sin h = 0. \tag{2.1}$$

In the notation of the previous paper

$$\Psi_+(x,w) = -\frac{i}{\sqrt{(2\pi)}} \int_0^\infty G(x,y,w)\psi(y)\,dy,$$

where
$$G(x,y,w) = \begin{cases} g(x,w)f(y,w) & (y < x), \\ f(x,w)g(y,w) & (y > x), \end{cases}$$

f and g being solutions of $(L-w)f = 0$. The corresponding functions for the finite interval $(0,a)$ are $\Psi_+(x,w,a)$, $G(x,y,w,a)$, $f(x,w)$, $g(x,w,a)$. Let $w_{n,a}$, $\psi_n(x,a)$, $c_{n,a}$ be the eigenvalues, eigenfunctions, and 'Fourier coefficients' of $\psi(x)$ for the interval $(0,a)$. Then

$$\Psi_+(x,w,a) = \frac{i}{\sqrt{(2\pi)}} \sum_{n=1}^{\infty} \frac{c_{n,a}\psi_n(x,a)}{w-w_{n,a}}. \tag{2.2}$$

Let w' be any number with positive imaginary part, and let $\gamma_{n,a}$ be the Fourier coefficient of $L\psi(x) - w'\psi(x)$. Then*

$$\gamma_{n,a} = (w_{n,a} - w')c_{n,a}.$$

Hence

$$|\Psi_+(x,w,a)| \leqslant \frac{1}{\sqrt{(2\pi)}v} \sum_{n=1}^{\infty} |c_{n,a}\psi_n(x,a)| = \frac{1}{\sqrt{(2\pi)}v} \sum_{n=1}^{\infty} \left|\frac{\gamma_{n,a}\psi_n(x,a)}{w_{n,a}-w'}\right|.$$

Now

$$\sum_{n=1}^{\infty} |\gamma_{n,a}|^2 \leqslant \int_0^a |L\psi(x) - w'\psi(x)|^2\,dx \leqslant \int_0^\infty |L\psi(x) - w'\psi(x)|^2\,dx.$$

Also $\psi_n(x,a)/(w_{n,a}-w')$ is the Fourier coefficient of $G(x,y,w',a)$. Hence

$$\sum_{n=1}^{\infty} \left|\frac{\psi_n(x,a)}{w_{n,a}-w'}\right|^2 \leqslant \int_0^a |G(x,y,w',a)|^2\,dy$$
$$= |g(x,w',a)|^2 \int_0^x |f(y,w')|^2\,dy + |f(x,w')|^2 \int_x^a |g(y,w',a)|^2\,dy.$$

* See e.g. § 8 of the previous paper.

Now $$g(x, w', a) = g(x, w') + \{l(w') - l_1(w')\} f(x, w'),$$

where $g(x, w')$ is $L^2(0, \infty)$, and either $f(x, w')$ is L^2 and $l \to l_1$, or

$$l - l_1 = O\left\{\int_0^a |f(x, w')|^2\, dx\right\}^{-1}.$$

Hence, as $a \to \infty$,

$$g(x, w', a) \to g(x, w'), \qquad \int_x^a |g(y, w', a)|^2\, dy \to \int_x^\infty |g(y, w')|^2\, dy,$$

$$\int_0^a |G(x, y, w', a)|^2\, dy \to \int_0^\infty |G(x, y, w')|^2\, dy.$$

Hence $$|\Psi_+(x, w, a)| \leqslant K/v,$$

where K is independent of w and a; and so

$$|\Psi_+(x, w)| \leqslant K/v. \tag{2.3}$$

Again $$\mathbf{R}\Psi_+(x, w, a) = \frac{v}{\sqrt{(2\pi)}} \sum_{n=1}^\infty \frac{c_{n,a}\psi_n(x, a)}{(u - w_{n,a})^2 + v^2}.$$

Hence

$$\int_{u_1}^{u_2} |\mathbf{R}\Psi_+(x, w, a)|\, du \leqslant \sqrt{(\tfrac{1}{2}\pi)} \sum_{n=1}^\infty |c_{n,a}\psi_n(x, a)| \leqslant M$$

as before, M being independent of u_1, u_2, v and a. Making $a \to \infty$, then $u_1 \to -\infty$, $u_2 \to \infty$, it follows that

$$\int_{-\infty}^\infty |\mathbf{R}\Psi_+(x, w)|\, du \leqslant M. \tag{2.4}$$

It therefore follows from § 1 that

$$\phi(x, \lambda) = \lim_{v \to 0} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\lambda \mathbf{R}\Psi_+(x, w)\, du \tag{2.5}$$

exists for all real values of λ, and $x \geqslant 0$, and is of bounded variation in $-\infty < \lambda < \infty$. Also

$$\int_0^\lambda \mathbf{R}\Psi_+(x, w, a)\, du = \frac{1}{\sqrt{(2\pi)}} \sum_{n=1}^\infty c_{n,a}\psi_n(x, a) \int_0^\lambda \frac{v\, du}{(u - w_{n,a})^2 + v^2}. \tag{2.6}$$

Hence, if $0 < a' < a$,

$$\int_0^{a'} \left\{\int_0^{\lambda} \mathbf{R}\Psi_+(x,w,a)\,du\right\}^2 dx \leqslant \int_0^{a} \left\{\int_0^{\lambda} \mathbf{R}\Psi_+(x,w,a)\,du\right\}^2 dx$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\left\{c_{n,a}\int_0^{\lambda}\frac{v\,du}{(u-w_{n,a})^2+v^2}\right\}^2 \leqslant \tfrac{1}{2}\pi\sum_{n=1}^{\infty}c_{n,a}^2 \leqslant \tfrac{1}{2}\pi\int_0^{\infty}\psi^2\,dx.$$

Making $a \to \infty$
$$\int_0^{a'}\left\{\int_0^{\lambda}\mathbf{R}\Psi_+(x,w)\,du\right\}^2 dx \leqslant \tfrac{1}{2}\pi\int_0^{\infty}\psi^2\,dx \tag{2.7}$$

for all a' and v. Making $v \to 0$, then $a' \to \infty$

$$\int_0^{\infty}\{\phi(x,\lambda)\}^2\,dx \leqslant \int_0^{\infty}\psi^2\,dx. \tag{2.8}$$

3. The function $\phi(x,\lambda)$ corresponds to a finite section of the Fourier series of $\psi(x)$. We next express $\psi(x)$ in terms of it. In addition to the previous hypotheses, let

$$W\{g(x,w),\psi(x)\} \to 0 \tag{3.1}$$

as $x \to \infty$ for every complex w. Then, by (5.1) of the previous paper,

$$\Psi_+(x,w) = \frac{i\psi(x)}{w\sqrt{(2\pi)}} - \frac{i}{w\sqrt{(2\pi)}}\int_0^{\infty}G(x,y,w)L\psi(y)\,dy. \tag{3.2}$$

Integrating along $(-R+i,\ R+i)$ and round the semicircle above it, we obtain

$$\psi(x) = \sqrt{\left(\frac{2}{\pi}\right)}\lim_{R\to\infty}\int_{-R+i}^{R+i}\Psi_+(x,w)\,dw.$$

The integral of the last term in (3.2) round the semicircle tends to zero, since it is $O(v^{-1}|w|^{-\frac{1}{2}})$, as in the particular case discussed in (III) of this series. Hence also

$$\psi(x) = \sqrt{\left(\frac{2}{\pi}\right)}\lim_{R\to\infty}\int_{-R}^{R}\mathbf{R}\Psi_+(x,u+i)\,du.$$

Now

$$\int_{-R+i}^{R+i}\Psi_+(x,w)\,dw = \left(\int_{-R+i}^{-R+i\delta} + \int_{-R+i\delta}^{R+i\delta} + \int_{R+i\delta}^{R+i}\right)\Psi_+(x,w)\,dw,$$

$$\int_{-R}^{R}\mathbf{R}\Psi_+(x,u+i)\,du = \int_{-R}^{R}\mathbf{R}\Psi_+(x,u+i\delta)\,du +$$
$$+\int_{\delta}^{1}\mathbf{I}\Psi_+(x,-R+iv)\,dv - \int_{\delta}^{1}\mathbf{I}\Psi_+(x,R+iv)\,dv.$$

By (1.2)
$$\Psi_+(x,w) = \frac{1}{i\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{d\phi(x,\lambda)}{\lambda-w},$$

$$\mathbf{I}\Psi_+(x,w) = -\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{\lambda-u}{(\lambda-u)^2+v^2}\,d\phi(x,\lambda),$$

$$\int_{\delta}^{1}\mathbf{I}\Psi_+(x,R+iv)\,dv = -\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\left(\arctan\frac{1}{\lambda-R}-\arctan\frac{\delta}{\lambda-R}\right)d\phi(x,\lambda)$$

$$= O\int_{-\infty}^{-\Delta}|d\phi(x,\lambda)|+O\int_{-\Delta}^{\Delta}\frac{|d\phi(x,\lambda)|}{R-\lambda}+O\int_{\Delta}^{\infty}|d\phi(x,\lambda)|,$$

which tends to zero as $R\to\infty$ (choosing first Δ and then R), uniformly with respect to δ. Hence on making $\delta\to 0$ and $R\to\infty$ we obtain

$$\psi(x) = \lim_{R\to\infty}\{\phi(x,R)-\phi(x,-R)\} = \int_{-\infty}^{\infty}d\phi(x,\lambda). \tag{3.3}$$

Again, (2.5) may be written

$$\phi(x,\lambda) =$$
$$\frac{1}{\pi}\lim_{v\to 0}\mathbf{I}\left\{\int_0^{\lambda}g(x,w)\,du\int_0^x f(y,w)\psi(y)\,dy+\int_0^{\lambda}f(x,w)\,du\int_x^{\infty}g(y,w)\psi(y)\,dy\right\}. \tag{3.4}$$

Since $g(x,w) = F(x,w)+l_1(w)f(x,w)$, and $F(x,u)$ and $f(x,u)$ are real for real u, this gives formally

$$\phi(x,\lambda) = \frac{1}{\pi}\int_0^{\lambda}\mathbf{I}l_1(u)f(x,u)\,du\int_0^{\infty}f(y,u)\psi(y)\,dy. \tag{3.5}$$

Hence (3.2) takes the form

$$\psi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\mathbf{I}l_1(u)f(x,u)\,du\int_0^{\infty}f(y,u)\psi(y)\,dy. \tag{3.6}$$

This will be valid in particular cases, but in general it has to be put into a form involving Stieltjes integrals.

4. We shall next prove some lemmas.

LEMMA α.
$$\int_0^\lambda \mathbf{I}l_1(w)\,du$$
is bounded for $0 \leqslant \lambda \leqslant \lambda_0$, $v \to 0$.

Consider the case of a finite interval with no singularities. Then
$$\int_0^a G(x,y,w,a)\psi_n(y,a)\,dy = \frac{\psi_n(x,a)}{w_{n,a}-w}.$$
Putting $x = 0$,
$$f(0,w)\int_0^a g(y,w,a)\psi_n(y,a)\,dy = \frac{\psi_n(0,a)}{w_{n,a}-w}.$$
Differentiating and putting $x = 0$,
$$f_x(0,w)\int_0^a g(y,w,a)\psi_n(y,a)\,dy = \frac{\psi_n'(0,a)}{w_{n,a}-w}.$$
Hence
$$\int_0^a g(y,w,a)\psi_n(y,a)\,dy = \frac{\psi_n(0,a)\sin h-\psi_n'(0,a)\cos h}{w_{n,a}-w} = \frac{C_{n,a}}{w_{n,a}-w}, \tag{4.1}$$
say. Thus, by the Parseval theorem,
$$\int_0^a |g(y,w,a)|^2\,dy = \sum_{n=1}^\infty \frac{C_{n,a}^2}{(w_{n,a}-u)^2+v^2}.$$
Hence
$$\int_0^\lambda du \int_0^a |g(y,w,a)|^2\,dy = \sum_{n=1}^\infty C_{n,a}^2 \int_0^\lambda \frac{du}{(w_{n,a}-u)^2+v^2}. \tag{4.2}$$
Now, for a fixed λ and $0 < v \leqslant 1$,
$$\int_0^\lambda \frac{v\,du}{(w_{n,a}-u)^2+v^2} = O\left(\frac{1}{w_{n,a}^2+1}\right).$$
Hence the right-hand side of (4.2) is
$$O\left(\frac{1}{v}\sum_{n=1}^\infty \frac{C_{n,a}^2}{w_{n,a}^2+1}\right) = O\left\{\frac{1}{v}\int_0^a |g(y,i,a)|^2\,dy\right\} = O\left\{\frac{1}{v}\int_0^\infty |g(y,i)|^2\,dy\right\}.$$

Making $a \to \infty$, we obtain

$$\int_0^\lambda du \int_0^\infty |g(y,w)|^2\, dy = O\left(\frac{1}{v}\right),$$

and the lemma follows, since*

$$\mathbf{I}l_1(w) = v \int_0^\infty |g(y,w)|^2\, dy.$$

LEMMA β. $$\int^\lambda |l_1(w)|\, du = O\left(\log\frac{1}{v}\right)$$

for $0 \leqslant \lambda \leqslant \lambda_0$, $v \to 0$.

Poisson's formula for the circle with centre $u+i$ and radius r gives

$$il_1(w) = \frac{1}{2\pi}\int_0^{2\pi} -\mathbf{I}l_1(u+i+re^{i\theta})\frac{u+i+re^{i\theta}+w}{u+i+re^{i\theta}-w}\, d\theta + i\mathbf{R}l_1(u+i).$$

Hence

$$|l_1(w)| < K\int_0^{2\pi} \frac{\mathbf{I}l_1(u+i+re^{i\theta})}{|re^{i\theta}+i-iv|}\, d\theta + K,$$

$$\int_0^\lambda |l_1(w)|\, du < K\int_0^{2\pi} \frac{d\theta}{|re^{i\theta}+i-iv|} + K$$

by Lemma α. Supposing $v \leqslant 1$, let $r = 1-\frac{1}{2}v$. Then according as $|\theta+\frac{3}{2}\pi|$ lies in the ranges $(0,v)$, $(v,\frac{1}{2}\pi)$, $(\frac{1}{2}\pi,\frac{3}{2}\pi)$, so

$$|re^{i\theta}+i-iv| \geqslant \tfrac{1}{2}v,\ A|\theta+\tfrac{3}{2}\pi|,\ A$$

respectively. Hence the right-hand side is

$$O\left\{\int_{|\theta+\frac{3}{2}\pi|\leqslant v} \frac{d\theta}{v}\right\} + O\left\{\int_{v\leqslant|\theta+\frac{3}{2}\pi|\leqslant\frac{1}{2}\pi} \frac{d\theta}{|\theta+\frac{3}{2}\pi|}\right\} + O(1) = O\left(\log\frac{1}{v}\right).$$

LEMMA ν. *There is a non-decreasing function $k(u)$ such that*

$$\lim_{v\to 0}\int_{\lambda_0}^{\lambda} \mathbf{I}g(x,w)\, du = \int_{\lambda_0}^{\lambda} f(x,u)\, dk(u)$$

except possibly for an enumerable set of values of λ_0 and λ. The left-hand side is bounded if λ and x lie in finite intervals.

* See the end of § 2 of the previous paper.

Consider the analytic function

$$\phi(w) = \int_{i}^{w} l_1(w')\,dw'.$$

By Lemma β,
$$\int_{0}^{\lambda} du \int_{0}^{1} |l_1(w)|\,dv$$

is finite. Hence
$$\int_{0}^{1} |l_1(w)|\,dv$$

is finite for almost all u, and so $\phi(w)$ tends to a finite limit as $v \to 0$ for almost all u. If u_0 is such a value, write

$$\mathbf{I}\phi(w) = \mathbf{I}\left(\int_{i}^{u_0+i} + \int_{u_0+i}^{u_0+iv} + \int_{u_0+iv}^{u+iv}\right) l_1(w')\,dw'.$$

Then it follows that

$$\lim_{v\to 0} \int_{u_0+iv}^{u+iv} \mathbf{I}l_1(w')\,dw' = k(u) \tag{4.3}$$

exists for almost all u. Since $\mathbf{I}l_1 \geqslant 0$, $k(u)$ is non-decreasing.

Actually the limit (4.3) exists wherever $k(u-0) = k(u+0)$. For, given ϵ, we can find δ and δ' such that

$$k(u+\delta) - k(u-\delta') < \epsilon.$$

Hence
$$\int_{u-\delta'}^{u} \mathbf{I}l_1(w')\,du \leqslant \int_{u-\delta'}^{u+\delta} \mathbf{I}l_1(w')\,du' < \epsilon$$

for $v < v_0$. Hence

$$\left| \int_{u_0}^{u} \mathbf{I}l_1(u'+iv)\,du' - \int_{u_0}^{u} \mathbf{I}l_1(u'+iv')\,du' \right| < 2\epsilon$$

if v and v' are sufficiently small. Hence $k(u)$ exists.

Suppose now that $k(u)$ is continuous at λ_0 and λ. We have

$$\int_{\lambda_0}^{\lambda} \mathbf{I}g(x,w)\,du = \int_{\lambda_0}^{\lambda} \{\mathbf{I}F(x,w) + \mathbf{R}l_1(w)\mathbf{I}f(x,w) + \mathbf{I}l_1(w)\mathbf{R}f(x,w)\}\,du.$$

Since $F(x,u)$ and $f(x,u)$ are real for real u, $\mathbf{I}F(x,w)$ and $\mathbf{I}f(x,w)$ are

207818B

$O(v)$, and the first two terms tend to 0 with v, by Lemma β. The third term is

$$[\mathbf{I}\phi(w)\mathbf{R}f(x,w)]_{\lambda_0}^{\lambda}-\int_{\lambda_0}^{\lambda}\mathbf{I}\phi(w)\mathbf{R}f_u(x,w)\,du$$

$$\to[\{k(u)+C\}f(x,u)]_{\lambda_0}^{\lambda}-\int_{\lambda_0}^{\lambda}\{k(u)+C\}f_u(x,u)\,du$$

$$=\int_{\lambda_0}^{\lambda}f(x,u)\,dk(u).$$

The process is justified since $\mathbf{I}\phi(w)$ is bounded, by Lemma α.

LEMMA δ. $$\int_0^\infty\left\{\int_0^\lambda \mathbf{I}g(x,w)\,du\right\}^2 dx$$

is bounded for $0\leqslant\lambda\leqslant\lambda_0$.

By (4.1)

$$\int_0^a\left\{\int_0^\lambda \mathbf{I}g(x,w,a)\,du\right\}\psi_n(y,a)\,dy=C_{n,a}\int_0^\lambda\frac{v\,du}{(w_{n,a}-u)^2+v^2}=O\left(\frac{|C_{n,a}|}{w_{n,a}^2+1}\right).$$

Hence the Parseval theorem gives

$$\int_0^a\left\{\int_0^\lambda \mathbf{I}g(x,w,a)\,du\right\}^2 dy=O\sum_{n=1}^{\infty}\frac{C_{n,a}^2}{1+w_{n,a}^2}=O\left\{\int_0^a|g(x,i,a)|^2\,dx\right\},$$

and the result follows as in Lemma α.

LEMMA ϵ. $$\int_0^\infty\left\{\int_0^\lambda \mathbf{R}g(x,w)\,du\right\}^2 dx=O\left(\log^2\frac{1}{v}\right)$$

for $0\leqslant\lambda\leqslant\lambda_0$.

The proof is similar to that of Lemma δ, except that we obtain

$$\int_0^\lambda\frac{w_{n,a}-u}{(w_{n,a}-u)^2+v^2}\,du$$

$$=O\left\{\int_{|w_{n,a}-u|\leqslant v}\frac{|w_{n,a}-u|}{v^2}\,du\right\}+O\left\{\int_{|w_{n,a}-u|>v}\frac{du}{|w_{n,a}-u|}\right\}=O\left(\log\frac{1}{v}\right)$$

uniformly in $w_{n,a}$. For large $|w_{n,a}|$ it is $O(|w_{n,a}|^{-1})$, so that it is

$$O\left\{(w_{n,a}^2+1)^{-\frac{1}{2}}\log\frac{1}{v}\right\}$$

for all $w_{n,a}$.

5. We can now prove the following theorem.*

Let $\psi(x)$ and $L\psi(x)$ belong to $L^2(0,\infty)$, and let $\psi(x)$ satisfy the boundary conditions (2.1) *and* (3.1). *Then*

$$\psi(x) = \int_{-\infty}^{\infty} f(x,u)\, d\xi(u), \tag{5.1}$$

where $\xi(u)$ is of bounded variation in $(-\infty,\infty)$; and

$$\xi(u) = \int_{0}^{\infty} \psi(y)\chi(y,u)\, dy, \tag{5.2}$$

where
$$\chi(y,\lambda) = \int_{0}^{\lambda} f(y,u)\, dk(u), \tag{5.3}$$

$k(u)$ being a non-decreasing function of u.

Since $\mathbf{I}F(x,w)$ and $\mathbf{I}f(x,w)$ are $O(v)$ as $v \to 0$,

$$\mathbf{I}\{g(x,w)f(y,w)-f(x,w)g(y,w)\}$$
$$= \mathbf{I}\{F(x,w)f(y,w)-f(x,w)F(y,w)\} = O(v)$$

uniformly for x, y and u in finite ranges. Hence (3.4) reduces to

$$\phi(x,\lambda) = \frac{1}{\pi}\lim_{v\to 0}\mathbf{I}\int_{0}^{\lambda} f(x,w)\, du \int_{0}^{\infty} g(y,w)\psi(y)\, dy.$$

By Lemma ϵ

$$\int_{0}^{\lambda} du \int_{0}^{\infty} \mathbf{R}g(y,w)\psi(y)\, dy = O\left(\log\frac{1}{v}\right).$$

Since $\mathbf{I}f(x,w) = O(v)$, $\mathbf{I}f_u(x,w) = O(v)$, it follows on integrating by parts that

$$\lim_{v\to 0}\int_{0}^{\lambda} \mathbf{I}f(x,w)\, du \int_{0}^{\infty} \mathbf{R}g(y,w)\psi(y)\, dy = 0.$$

Hence
$$\phi(x,\lambda) = \frac{1}{\pi}\lim_{v\to 0}\int_{0}^{\lambda} \mathbf{R}f(x,w)\, du \int_{0}^{\infty} \mathbf{I}g(y,w)\psi(y)\, dy.$$

Let
$$\xi(\lambda,v) = \int_{\lambda_0}^{\lambda} du \int_{0}^{\infty} \mathbf{I}g(y,w)\psi(y)\, dy.$$

* Weyl, loc. cit., Satz 7.

Then

$$\phi(x,\lambda)-\phi(x,\lambda_0) = \frac{1}{\pi}\lim_{v\to 0}\Big\{\xi(\lambda,v)\mathbf{R}f(x,\lambda+iv)-$$

$$-\xi(\lambda_0,v)\mathbf{R}f(x,\lambda_0+iv)-\int_{\lambda_0}^{\lambda}\xi(u,v)\mathbf{R}f_u(x,u+iv)\,du\Big\}.$$

By Lemmas γ and δ, and Lemma γ of the previous paper,

$$\lim_{v\to 0}\xi(\lambda,v) = \int_0^\infty \psi(y)\chi(y,\lambda)\,dy = \xi(\lambda),$$

where

$$\chi(y,\lambda) = \int_{\lambda_0}^{\lambda} f(y,u)\,dk(u),$$

and λ_0 and λ are points of continuity of $k(u)$. Hence

$$\phi(x,\lambda)-\phi(x,\lambda_0) = \frac{1}{\pi}\Big\{\xi(\lambda)f(x,\lambda)-\xi(\lambda_0)f(x,\lambda_0)-\int_{\lambda_0}^{\lambda}\xi(u)f_u(x,u)\,du\Big\}. \tag{5.4}$$

Since $f(0,u) = \sin h$, $f_x(0,u) = \cos h$, $f_u(0,u) = 0$, $f_{xu}(0,u) = 0$, we obtain on putting $x = 0$

$$\phi(0,\lambda)-\phi(0,\lambda_0) = \frac{1}{\pi}\{\xi(\lambda)-\xi(\lambda_0)\}\sin h.$$

If $\sin h \neq 0$, it follows from § 2 that $\xi(\lambda)$ is of bounded variation in $(-\infty,\infty)$. If $\sin h = 0$, we obtain

$$\phi_x(0,\lambda)-\phi_x(0,\lambda_0) = \pm\frac{1}{\pi}\{\xi(\lambda)-\xi(\lambda_0)\}.$$

Now $\phi_x(0,\lambda)$ is also of bounded variation in $(-\infty,\infty)$; for

$$\pm\psi_n'(0,a)/(w'-w_{n,a})$$

is the 'Fourier coefficient' of $g(y,w')$, and we obtain as in § 2

$$\int_{-\infty}^{\infty} |\mathbf{R}\Psi_{+,x}(0,w)|\,du \leqslant M.$$

It follows that in any case $\xi(\lambda)$ is of bounded variation, and we may write (5.4) as

$$\phi(x,\lambda)-\phi(x,\lambda_0) = \frac{1}{\pi}\int_{\lambda_0}^{\lambda} f(x,u)\,d\xi(u) \tag{5.5}$$

since $\chi(y,u)$ and so also $\xi(u)$ are continuous at λ_0 and λ.

Let us define $k(u)$ at discontinuities by

$$k(u) = \tfrac{1}{2}\{k(u+0)+k(u-0)\}.$$

Then $\chi(y,u)$ and $\xi(u)$ satisfy similar relations. Since, by § 1, $\phi(x,\lambda)$ also satisfies this relation, (5.5) holds for all λ_0 and λ. Taking $\lambda_0 = 0$, the theorem follows.

6. The representation of $\psi(x)$ has the following orthogonal property.

If $0 \leqslant \lambda < \lambda' < \mu < \mu'$, *or* $0 \leqslant \mu < \mu' < \lambda < \lambda'$,

$$\int_0^\infty \{\phi(x,\lambda)-\phi(x,\lambda')\}\{\phi(x,\mu)-\phi(x,\mu')\}\,dx = 0.$$

By (2.6) and the Parseval theorem

$$\int_0^a dx \int_\lambda^{\lambda'} \mathbf{R}\Psi_+(x,w,a)\,du \int_\mu^{\mu'} \mathbf{R}\Psi_+(x,w',a)\,du'$$

$$= \frac{1}{2\pi}\sum_{n=1}^\infty c_{n,a}^2 \int_\lambda^{\lambda'} \frac{v\,du}{(u-w_{n,a})^2+v^2} \int_\mu^{\mu'} \frac{v'\,du'}{(u'-w_{n,a})^2+v'^2}.$$

Suppose that $\lambda < \lambda' < \mu < \mu'$. Then for every $w_{n,a}$ either

$$|u-w_{n,a}| \geqslant \tfrac{1}{2}(\mu-\lambda') \quad \text{or} \quad |u'-w_{n,a}| \geqslant \tfrac{1}{2}(\mu-\lambda').$$

Hence the right-hand side does not exceed

$$\frac{1}{2}\sum_{n=1}^\infty c_{n,a}^2 \max\left\{\frac{4(\lambda'-\lambda)v}{(\mu-\lambda')^2}, \frac{4(\mu'-\mu)v'}{(\mu-\lambda')^2}\right\}$$

$$= \frac{2\max\{(\lambda'-\lambda)v, (\mu'-\mu)v'\}}{(\mu-\lambda')^2}\int_0^\infty \psi^2\,dx.$$

Now, if $\psi(x) = 0$ for $x > \Delta$,

$$\Psi_+(x,w,a)-\Psi_+(x,w) = -\frac{i}{\sqrt{(2\pi)}}(l-l_1)f(x,w)\int_0^\Delta f(y,w)\psi(y)\,dy$$

for $a > \Delta$. Hence, as $a \to \infty$,

$$\int_0^a |\Psi_+(x,w,a)-\Psi_+(x,w)|^2\,dx$$

$$\leqslant \frac{|l-l_1|^2}{2\pi}\int_0^a |f(x,w)|^2\,dx \left|\int_0^\Delta f(y,w)\psi(y)\,dy\right|^2 \to 0$$

by (2.12) of the previous paper. This result also holds for any ψ of L^2, since we can replace Δ by ∞ in the left-hand side with error $O\Big(\int\limits_{\Delta}^{\infty} \psi^2\,dx\Big)$. Hence, on making $a \to \infty$ in the above inequality, we obtain

$$\left|\int\limits_0^\infty dx \int\limits_\lambda^{\lambda'} \mathbf{R}\Psi_+(x,w)\,du \int\limits_\mu^{\mu'} \mathbf{R}\Psi_+(x,w')\,du'\right| \leqslant \frac{2\max\{(\lambda'-\lambda)v,(\mu'-\mu)v'\}}{(\mu-\lambda')^2} \int\limits_0^\infty \psi^2\,dx.$$

Making $v \to 0$, $v' \to 0$, and using Lemma γ of the previous paper, the result follows.

7. We now turn to the general case* of all functions $\psi(x)$ of $L^2(0,\infty)$. We first require the following result.

Under the special conditions of the above theorem, $\phi(x,\lambda)-\phi(x,-\lambda)$ *converges in mean to* $\psi(x)$ *over* $(0,\infty)$.

By the above theorem, $\phi(x,\lambda)-\phi(x,-\lambda)$ tends to $\psi(x)$ uniformly over any finite interval. By (2.8), with the interval $(0,\lambda)$ replaced by $(-\lambda,\lambda)$,

$$\int\limits_0^\infty \{\phi(x,\lambda)-\phi(x,-\lambda)\}^2\,dx \leqslant \int\limits_0^\infty \{\psi(x)\}^2\,dx.$$

Hence by Lemma γ of the previous paper

$$\lim_{\lambda\to\infty} \int\limits_0^\infty \psi(x)\{\psi(x)-\phi(x,\lambda)+\phi(x,-\lambda)\}\,dx = 0.$$

Hence

$$\begin{aligned}\overline{\lim_{\lambda\to\infty}} \int\limits_0^\infty \{\psi(x)-\phi(x,\lambda)+\phi(x,-\lambda)\}^2\,dx &= \overline{\lim_{\lambda\to\infty}}\Big[\int\limits_0^\infty \{\psi(x)\}^2\,dx - 2\int\limits_0^\infty \psi(x)\{\phi(x,\lambda)-\phi(x,-\lambda)\}\,dx + \\ &\qquad + \int\limits_0^\infty \{\phi(x,\lambda)-\phi(x,-\lambda)\}^2\,dx\Big] \\ &= \overline{\lim_{\lambda\to\infty}}\Big[-\int\limits_0^\infty \{\psi(x)\}^2\,dx + \int\limits_0^\infty \{\phi(x,\lambda)-\phi(x,-\lambda)\}^2\,dx\Big] \leqslant 0.\end{aligned}$$

Since it is clearly not negative, the result follows.

* Not considered by Weyl.

Now let $\psi(x)$ be any function of $L^2(0,\infty)$. Then there is a sequence $\psi^{(n)}(x)$ of functions of the special class such that $\psi(x) = \text{l.i.m.}\,\psi^{(n)}(x)$. Let $\phi^{(n)}(x,\lambda)$ correspond to $\psi^{(n)}(x)$ as the above $\phi(x,\lambda)$ corresponds to $\psi(x)$. Then, by (2.8),

$$\int_0^\infty \{\phi^{(m)}(x,\lambda)-\phi^{(n)}(x,\lambda)\}^2\,dx \leqslant \int_0^\infty \{\psi^{(m)}(x)-\psi^{(n)}(x)\}^2\,dx,$$

which tends to zero as m and n tend to infinity. Hence $\phi^{(n)}(x,\lambda)$ converges in mean, to $\phi(x,\lambda)$ say. This $\phi(x,\lambda)$ is $L^2(0,\infty)$, but is not necessarily of bounded variation as a function of λ.

We have

$$\int^\infty \{\psi(x)-\phi(x,\lambda)+\phi(x,-\lambda)\}^2\,dx < A\int_0^\infty \{\psi(x)-\psi^{(n)}(x)\}^2\,dx +$$
$$+A\int_0^\infty \{\psi^{(n)}(x)-\phi^{(n)}(x,\lambda)+\phi^{(n)}(x,-\lambda)\}^2\,dx +$$
$$+A\int_0^\infty \{\phi^{(n)}(x,\lambda)-\phi^{(n)}(x,-\lambda)-\phi(x,\lambda)+\phi(x,-\lambda)\}^2\,dx,$$

which can be made arbitrarily small for $\lambda > \lambda_0$ by choosing first n and then λ_0. Hence $\phi(x,\lambda)-\phi(x,-\lambda)$ converges in mean to $\psi(x)$.

Defining $\xi(u)$ by (5.2), $\xi^{(n)}(u) \to \xi(u)$ boundedly over a finite range. Hence as $n \to \infty$

$$\phi^{(n)}(x,\lambda) = \frac{1}{\pi}\left\{\xi^{(n)}(\lambda)f(x,\lambda)-\int_0^\lambda \xi^{(n)}(u)f_u(x,u)\,du\right\}$$
$$\to \frac{1}{\pi}\left\{\xi(\lambda)f(x,\lambda)-\int_0^\lambda \xi(u)f_u(x,u)\,du\right\}$$

for every x and λ. Hence we have the following theorem.

Let $\psi(x)$ be any function of $L^2(0,\infty)$, Then

$$\psi(x) = \text{l.i.m.}\{\phi(x,\lambda)-\phi(x,-\lambda)\}, \tag{7.1}$$

where
$$\phi(x,\lambda) = \frac{1}{\pi}\left\{\xi(\lambda)f(x,\lambda)-\int_0^\lambda \xi(u)f_u(x,u)\,du\right\}, \tag{7.2}$$

$\xi(u)$ being defined by (5.2) and $\chi(y,\lambda)$ by (5.3).

Again, let
$$\sigma(\lambda) = \int_0^\infty \psi(x)\phi(x,\lambda)\,dx. \tag{7.3}$$

We have

$$\int_0^\infty \psi(x)\mathbf{R}\Psi_+(x,w)\,dx = \sqrt{(2\pi)}v\int_0^\infty |\Psi_+(x,w)|^2\,dx \geqslant 0$$

for $v > 0$. Hence

$$\int_0^\infty \psi(x)\,dx \int_0^\lambda \mathbf{R}\Psi_+(x,w)\,du$$

is a non-decreasing function of λ for every positive v. By (2.7) it is a bounded function of λ and v; by (2.5), (2.7), and Lemma γ of the previous paper, it tends to $\sigma(\lambda)$ as $v \to 0$. Hence $\sigma(\lambda)$ is non-decreasing and bounded in $(-\infty,\infty)$. Also

$$\begin{aligned}\int_{-\infty}^\infty d\sigma(\lambda) &= \lim_{\lambda\to\infty}\{\sigma(\lambda)-\sigma(-\lambda)\}\\ &= \lim_{\lambda\to\infty}\int_0^\infty \psi(x)\{\phi(x,\lambda)-\phi(x,-\lambda)\}\,dx\\ &= \int_0^\infty \{\psi(x)\}^2\,dx. \qquad (7.4)\end{aligned}$$

This formula contains the Parseval theorem of the ordinary cases. Suppose, for example, that the only singularities of $\Psi_+(x,w)$ are poles, so that the expansion is a series. Then, as in § 6 of the previous paper,

$$\sigma(\lambda) = \sum_{0\leqslant w_n\leqslant\lambda}{}' c_n^2,$$

where the c_n are the Fourier coefficients, and the dash indicates that terms with $w_n = 0$ or λ are halved. Hence (7.4) becomes

$$\sum_{n=1}^\infty c_n^2 = \int_0^\infty \{\psi(x)\}^2\,dx.$$

Suppose on the other hand that $k(u)$ is absolutely continuous. Then we have formally

$$\chi(y,\lambda) = \int_0^\infty f(y,u)k'(u)\,du,$$

$$\xi'(u) = \int_0^\infty \psi(y)f(y,u)k'(u)\,dy,$$

and

$$\phi(x,\lambda) = \frac{1}{\pi}\int_0^\lambda f(x,u)\xi'(u)\,du.$$

Hence

$$\sigma(\lambda) = \frac{1}{\pi}\int\limits_0^\lambda \xi'(u)\,du \int\limits_0^\infty \psi(x)f(x,u)\,dx = \frac{1}{\pi}\int\limits_0^\lambda \frac{\{\xi'(u)\}^2}{k'(u)}\,du.$$

Hence (7.4) becomes

$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{\{\xi'(u)\}^2}{k'(u)}\,du = \int\limits_0^\infty \{\psi(x)\}^2\,dx.$$

8. Example. As an example of the above analysis, let $L \equiv d^2/dx^2$. Then

$$f(x,w) = \frac{1}{2}\left(\sin h - \frac{\cos h}{\sqrt{w}}\right)e^{x\sqrt{w}} + \frac{1}{2}\left(\sin h + \frac{\cos h}{\sqrt{w}}\right)e^{-x\sqrt{w}},$$

$$F(x,w) = \frac{1}{2}\left(\cos h + \frac{\sin h}{\sqrt{w}}\right)e^{x\sqrt{w}} + \frac{1}{2}\left(\cos h - \frac{\sin h}{\sqrt{w}}\right)e^{-x\sqrt{w}}.$$

We have to determine $l_1(w)$ so that $g(x,w) = F(x,w)+l_1(w)f(x,w)$ is $L^2(0,\infty)$ for $\mathbf{I}(w) > 0$; since $|e^{x\sqrt{w}}|$ is large at infinity the only such function is

$$l_1(w) = \frac{\sin h + \sqrt{w}\cos h}{\cos h - \sqrt{w}\sin h}.$$

As $v \to 0$, $$\lim \mathbf{I}l_1(w) = \begin{cases} 0 & (u>0), \\ \dfrac{\sqrt{(-u)}}{\cos^2 h - u\sin^2 h} & (u<0). \end{cases}$$

Hence $$\chi(y,\lambda) = \begin{cases} 0 & (\lambda > 0), \\ \displaystyle\int\limits_\lambda^0 f(y,u)\frac{\sqrt{(-u)}}{\cos^2 h - u\sin^2 h}\,du & (\lambda < 0). \end{cases}$$

The resulting integral formula is

$$\psi(x) = \int\limits_0^\infty \frac{\sqrt{u}}{\cos^2 h + u\sin^2 h} f(x,-u)\,du \int\limits_0^\infty f(y,-u)\psi(y)\,dy,$$

where $$f(x,-u) = \sin h \cos x\sqrt{u} - \frac{\cos h \sin x\sqrt{u}}{\sqrt{u}}.$$

ON THE RESULTANT OF TWO FUNCTIONS

By M. M. CRUM (*Oxford*)

[Received 31 March 1941]

THE object of this paper is to prove the following theorem:

THEOREM. *Let $f(x)$, $g(x)$ belong to $L(0,c)$.*

Let
$$h(x) = \int_0^x f(y)g(x-y)\,dy.$$

Then, if $h(x)$ is null in $(0,c)$, $f(x)$ is null in $(0,a)$ and $g(x)$ is null in $(0,b)$ where $a+b = c$.

The theorem is proved by Titchmarsh† by means of an importart but difficult theorem on certain integral functions; the present proof is intended to be simpler.

We require three lemmas. The first is a standard Phragmèn-Lindelöf theorem, stated here for reference in the form in which it is needed.

LEMMA A. *Let $F(w)$ be regular in and on the boundary of R, where R lies inside the region*
$$\lambda \leqslant \operatorname{am}(w-w_0) \leqslant \mu.$$
Let $F(w)$ be of order‡ less than $\pi/(\mu-\lambda)$ in R. Let $|F(w)| \leqslant M$ on the boundary of R. Then $|F(w)| \leqslant M$ in R.

The proof is the same as that of the main Phragmèn-Lindelöf theorem.

LEMMA B. *Let $F(w)$ be regular in and on the boundary of R, where R lies inside $R_1+R_2+R_3$, and R_1 is the region $\lambda_1 \leqslant \operatorname{am} w \leqslant \mu_1$, R_2 is the region $\lambda_2 \leqslant \operatorname{am} w \leqslant \mu_2$, R_3 is the region $|w| < \rho$. Let $F(w)$ be of order less than $\pi/(\mu_1-\lambda_1)$ in RR_1, and of order less than $\pi/(\mu_2-\lambda_2)$ in RR_2. Let $|F(w)| \leqslant M$ on the boundary of R.*

Then $|F(w)| \leqslant M$ in R.

Let M_1, M_2 be the upper bounds of $|F(w)|$ on the boundaries of $R(R_1+R_3)$, $R(R_2+R_3)$. By Lemma A
$$|F(w)| \leqslant M_1 \text{ in } R(R_1+R_3), \qquad |F(w)| \leqslant M_2 \text{ in } R(R_2+R_3).$$
Hence $F(w)$ is bounded in R.

† E. C. Titchmarsh, *Proc. London Math. Soc.* (2) 25 (1926), 283–302, and *Fourier Integrals* (Oxford, 1937), 323–8.

‡ $F(w)$ is of order ρ if it is of the form $\exp(|w|^{\rho+\epsilon})$ for every positive ϵ.

Let w_0 be distant δ from any point of R. If there is in the first instance no point outside R, we can supply one by a conformal transformation. Let

$$F_\epsilon(w) = \delta^\epsilon(w-w_0)^{-\epsilon}F(w).$$

Then $|F_\epsilon(w)| \leqslant M$ on the boundaries of R, and $F_\epsilon(w)$ is $o(1)$ as $|w| \to \infty$ in R. Hence $|F_\epsilon(w)| \leqslant M$ in R, or, letting $\epsilon \to 0$, we have $|F(w)| \leqslant M$ in R.

Lemma C. *Let $F(w)$ and $G(w)$ be analytic functions of $w = u+iv$, regular for $u \geqslant 0$. Let $|F(w)| \leqslant 1$, and $|G(w)| \leqslant 1$ for $u = 0$. Let F and G be of order* 1 *for $-\frac{1}{2}\pi \leqslant \operatorname{am} w \leqslant \frac{1}{2}\pi$, and let neither be identically zero. Let $|F(w)G(w)| \leqslant e^{-cu}$ for $u \geqslant 0$. Then there exist a and b such that $a+b = c$ and*

$$|F(w)| \leqslant e^{-au}, \qquad |G(w)| \leqslant e^{-bu} \textit{ for } u \geqslant 0.$$

Let $\qquad F^*(w) = F(w)\overline{F(\bar{w})}, \qquad G^*(w) = G(w)\overline{G(\bar{w})}.$

Then the same inequalities hold for F^* and G^* as for F and G, with $2c$ replacing c.

We consider the regions R_1, R_2, in which respectively

$$|F^*(w)e^{2\alpha w}| \geqslant 1+\epsilon, \qquad |G^*(w)e^{2\beta w}| \geqslant 1+\epsilon,$$

where α and β are any fixed real numbers whose sum is c, and ϵ is positive. We wish to show that either R_1 or R_2 is empty.

Suppose neither is empty. Let w_1 be in R_1, w_2 in R_2. Let W_1, W_2 be the connected regions of R_1, R_2 including w_1 and w_2. Suppose that, for some ρ, W_1 contains no real point u_1, where $u_1 > \rho$. Then, by Lemma B,

$$|F^*(w)e^{2\alpha w}| \leqslant 1+\epsilon \quad \text{in } W_1.$$

Since, by definition of W_1, the reversed inequality holds, we must have $F^*(w)e^{2\alpha w} = (1+\epsilon)e^{i\omega}$, a constant, and this contradicts $|F^*(iv)| \leqslant 1$. Hence there are arbitrarily large real values of w in W_1, and so also in W_2.

Let u_1 be in W_1. Then we can find u_2 in W_2 where $u_2 > u_1$, and u_3 in W_1 with $u_3 > u_2$. Since W_1 is a connected region we can connect u_1 and u_3 by a curve γ in W_1. Since $F^*(w)$ is real for real w, the reflection $\bar{\gamma}$ of γ in the real axis is also in W_1. On these curves we have

$$|G^*(w)e^{2\beta w}| \leqslant 1/(1+\epsilon).$$

Hence, by the maximum-modulus theorem,

$$|G^*(u_2)e^{2\beta u_2}| \leqslant 1/(1+\epsilon),$$

which contradicts the definition of u_2. Hence either R_1 or R_2 is empty.

Suppose that for some w_1 and w_2

$$|F^*(w_1)| > e^{-2\alpha u_1}, \qquad |G^*(w_2)| > e^{-2\beta u_2}.$$

Then, for some positive ϵ,

$$|F^*(w_1)| \geqslant (1+\epsilon)e^{-2\alpha u_1}, \qquad |G^*(w_2)| \geqslant (1+\epsilon)e^{-2\beta u_2}.$$

Since we have shown that this cannot be so, it follows that

either $\qquad |F^*(w)| \leqslant e^{-2\alpha u}$ for all $u \geqslant 0$

or $\qquad |G^*(w)| \leqslant e^{-2\beta u}$ for all $u \geqslant 0$.

Let a be the upper bound of α for which the first holds. If it held for all α, F^* and F would be identically zero; if it held for no α, the second would hold for all β, and G^* and G would be identically zero. Hence a is finite. Then, for $\epsilon > 0$, $|F^*(w)| \leqslant e^{-2(a-\epsilon)u}$ for all $u \geqslant 0$. Letting $\epsilon \to 0$, we have $|F^*(w)| \leqslant e^{-2au}$ for all $u \geqslant 0$.

Also, for $\epsilon > 0$, we can find w_1 such that

$$|F^*(w_1)| > e^{-2(a+\epsilon)u_1},$$

and so for all $u > 0$, if $b = c-a$,

$$|G^*(w)| \leqslant e^{-2(b-\epsilon)u}.$$

Letting $\epsilon \to 0$, $|G^*(w)| \leqslant e^{-2bu}$ for all $u \geqslant 0$.

Now for real w

$$|F(u)| = |F^*(u)|^{\frac{1}{2}} \leqslant e^{-au}, \qquad |G(u)| = |G^*(u)|^{\frac{1}{2}} \leqslant e^{-bu}.$$

Applying Lemma A to $F(w)e^{aw}$ and $G(w)e^{bw}$ we have, for all $u \geqslant 0$,

$$|F(w)| \leqslant e^{-au}, \qquad |G(w)| \leqslant e^{-bu},$$

where $a+b = c$.

Proof of the theorem. We may suppose that

$$\int_0^c |f(x)|\, dx \leqslant 1, \qquad \int_0^c |g(x)|\, dx \leqslant 1,$$

and that f and g are null for $x < 0$ and $x > c$. Then $h(x)$ is null for $x < c$ and $x > 2c$; and

$$\int_c^{2c} |h(x)|\, dx = \int_c^{2c} dx \left| \int_0^c f(y)g(x-y)\, dy \right|$$

$$\leqslant \int_0^c |f(y)|\, dy \int_0^{2c} |g(x-y)|\, dx = \int_0^c |f(y)|\, dy \int_0^c |g(x)|\, dx \leqslant 1.$$

Let $$F(w) = \int_0^c f(x)e^{-xw}\,dx, \qquad G(w) = \int_0^c g(x)e^{-xw}\,dx.$$

Then $$|F(iv)| \leqslant 1, \qquad |G(iv)| \leqslant 1|,$$

and

$$H(w) = \int_0^{2c} h(x)e^{-xw}\,dx = \int_0^{2c} e^{-xw}\,dx \int_0^c f(y)g(x-y)\,dy$$

$$= \int_0^c f(y)\,dy \int_0^{2c} e^{-xw}g(x-y)\,dx$$

$$= \int_0^c f(y)e^{-yw}\,dy \int_0^c g(x)e^{-xw}\,dx = F(w)G(w).$$

Hence

$$|F(w)G(w)| = \left|\int_c^{2c} h(x)e^{-xw}\,dx\right| \leqslant |e^{-cw}| \int_c^{2c} |h(x)e^{-(x-c)w}|\,dx \leqslant e^{-cu},$$

if $u \geqslant 0$.

Hence by Lemma C either $F(w) \equiv 0$ or $G(w) \equiv 0$ or there exist a and b such that $a+b = c$, and $|F(w)| \leqslant e^{-au}$, $|G(w)| \leqslant e^{-bu}$ for $u \geqslant 0$.

Hence* in any case $f(x)$ is null in $(0, a)$ and $g(x)$ is null in $(0, b)$ where $a+b = c$.

† Titchmarsh, *Fourier Integrals*, 325.

EXPANSIONS OF APPELL'S DOUBLE HYPERGEOMETRIC FUNCTIONS (II)

By J. L. BURCHNALL (*Durham*) and T. W. CHAUNDY (*Oxford*)

[Received 27 April 1941]

IN a previous paper under this title (**5**) we obtained numerous formulae involving the elementary hypergeometric functions of Appell's type. From these a large number of other formulae are deducible as corollaries. It would be tedious to put all these on record, and we collect here typical examples that should be sufficiently representative. For the most part these formulae appear as special cases of corresponding formulae of (**5**) arising from special values of the arguments or the parameters; in particular, the bulk of them relate to the double confluent hypergeometric functions discussed by Humbert. We begin, however, with some extensions of our main formulae.

1. Extensions to functions of higher order

Appell's functions have been defined in terms of products of simple hypergeometric functions, with one or more parameters the same, in a way that we have written symbolically* as

$$F^{(1)}[a;b,b';c;x,y] = \nabla(a)\Delta(c)F(a,b;c;x)F(a,b';c;y), \tag{1}$$

$$F^{(2)}[a;b,b';c,c';x,y] = \nabla(a)F(a,b;c;x)F(a,b';c';y), \tag{2}$$

$$F^{(3)}[a,a';b,b';c;x,y] = \Delta(c)F(a,b;c;x)F(a',b';c;y), \tag{3}$$

$$F^{(4)}[a,b;c,c';x,y] = \nabla(a)\nabla(b)F(a,b;c;x)F(a,b;c';y), \tag{4}$$

where ∇, Δ are symbolic operators defined as

$$\nabla(h) \equiv \frac{\Gamma(h)\Gamma(\delta+\delta'+h)}{\Gamma(\delta+h)\Gamma(\delta'+h)}, \qquad \Delta(h) \equiv \frac{\Gamma(\delta+h)\Gamma(\delta'+h)}{\Gamma(h)\Gamma(\delta+\delta'+h)}. \tag{5}$$

If now we extend these formulae so that on the right the parameters of ∇, Δ and the F's are all different, we get on the left double hypergeometric functions of higher order. Thus

$$\nabla(h)\Delta(k)F(a,b;c;x)F(a',b';c';y)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(h)_{m+n}(k)_m(a)_m(b)_m(k)_n(a')_n(b')_n}{m!\,n!\,(k)_{m+n}(h)_m(c)_m(h)_n(c')_n} x^m y^n$$

$$\equiv F\begin{bmatrix} h:k,a,b;k,a',b'; \\ k:\ h,c;\quad h,c'; \end{bmatrix} x,y, \tag{6}$$

* (**5**) § 1.

$$\nabla(h)F(a,b;c;x)F(a',b';c';y)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(h)_{m+n}(a)_m(b)_m(a')_n(b')_n}{m!\,n!\,(h)_m(c)_m(h)_n(c')_n}x^m y^n$$

$$\equiv F\begin{bmatrix} h{:}\,a,b;a',b'; \\ h,c;\;h,c'; \end{bmatrix} x,y\Bigg], \tag{7}$$

$$\Delta(k)F(a,b;c;x)F(a',b';c';y)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(k)_m(a)_m(b)_m(k)_n(a')_n(b')_n}{m!\,n!\,(k)_{m+n}(c)_m(c')_n}x^m y^n, \tag{8}$$

$$\equiv F\left[\begin{matrix} & k,a,b;k,a',b'; \\ k{:} & c; \qquad c'; \end{matrix}\; x,y\right], \tag{9}$$

$$\nabla(h)\nabla(k)F(a,b;c;x)F(a',b';c';y)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(h)_{m+n}(k)_{m+n}(a)_m(b)_m(a')_n(b')_n}{m!\,n!\,(h)_m(k)_m(c)_m(h)_n(k)_n(c')_n}x^m y^n$$

$$\equiv F\left[\begin{matrix} h,k{:}\; a,b;\; a',b'; \\ h,k,c;h,k,c'; \end{matrix}\; x,y\right], \tag{10}$$

in a notation that seems more economical than that suggested by Appell and Kampé de Fériet.*

The expansions for ∇, Δ, $\nabla\Delta$ given in (5) § 2 are still valid, and we obtain expansions analogous to those of (5) § 3. For example,

$$F\left[\begin{matrix} h{:}\,a,b;a',b'; \\ h,c;\;h,c'; \end{matrix}\; x,y\right]$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(a')_r(b')_r}{r!\,(h)_r(c)_r(c')_r}\times$$

$$\times x^r y^r F(a+r,b+r;c+r;x)F(a'+r,b'+r;c'+r;y), \tag{11}$$

$$F\left[\begin{matrix} & k,a,b;k,a',b'; \\ k{:} & c; \qquad c'; \end{matrix}\; x,y\right]$$

$$=\sum_{r=0}^{\infty}(-)^r\frac{\{(k)_r\}^2(a)_r(b)_r(a')_r(b')_r}{r!\,(k+r-1)_r(k)_{2r}(c)_r(c')_r}x^r y^r {}_3F_2\left[\begin{matrix} k+r,a+r,b+r; \\ k+2r,c+r \end{matrix}\; x\right]\times$$

$$\times {}_3F_2\left[\begin{matrix} k+r,a'+r,b'+r; \\ k+2r,c'+r \end{matrix}\; y\right], \tag{12}$$

* (1) 150.

$$F\begin{bmatrix} h\colon k,a,b;k,a',b'; \\ k\colon h,c; \quad h,c'; \end{bmatrix} x,y$$

$$= \sum_{r=0}^{\infty} \frac{\{(k_r)\}^2(k-h)_r(a)_r(b)_r(a')_r(b')_r}{r'(k+r-1)_r(k)_{2r}(h)_r(c)_r(c')_r} x^r y^r {}_3F_2\begin{bmatrix} k+r,a+r,b+r; \\ k+2r,c+r \end{bmatrix} x \Bigg] \times$$

$$\times {}_3F_2\begin{bmatrix} k+r,a'+r,b'+r; \\ k+2r,c'+r \end{bmatrix} y \Bigg] \quad (13)$$

are analogues of (**5**) (26), (28), (30). To recover the original formulae as particular cases we write $h = a' = a$ in (11) to give (**5**) (26) and so on.

If in (13) we put $h = \frac{1}{2}k$, we get

$$F\begin{bmatrix} \frac{1}{2}k\colon k,a,b;k,a',b'; \\ k\colon \frac{1}{2}k,c; \quad \frac{1}{2}k,c'; \end{bmatrix} x,y$$

$$= \sum_{r=0}^{\infty} \frac{\{(k)_r\}^2(a)_r(b)_r(a')_r(b')_r}{r!\,(k+r-1)_r(k)_{2r}(c)_r(c')_r} x^r y^r {}_3F_2\begin{bmatrix} k+r,a+r,b+r; \\ k+2r,c+r \end{bmatrix} x \Bigg] \times$$

$$\times {}_3F_2\begin{bmatrix} k+r,a'+r,b'+r; \\ k+2r,c'+r \end{bmatrix} y \Bigg], \quad (14)$$

where the terms in the summation differ only in sign from those in (12).

2. 'Duplication' and other formulae for ${}_2F_1$

We collect here some formulae involving only the simple hypergeometric function, of which (21) and (24), where the argument is respectively doubled and squared, may be regarded rather loosely as 'duplication formulae'.

We note firstly that, since

$$F^{(1)}[a;b,b';c;x,x] = F(a,b+b';c;x), \quad (15)$$

we have from (**5**) (30)

$$F(a,b+b';c;x) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(b')_r(c-a)_r}{r!\,(c+r-1)_r(c)_{2r}} x^{2r} \times$$

$$\times F(a+r,b+r;c+2r;x)F(a+r,b'+r;c+2r;x), \quad (16)$$

which we can regard as an addition-formula in the arguments b, b'. From (**5**) (31) we derive the inverse formula

$$F(a,b;c;x)F(a,b';c;x)$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(b')_r(c-a)_r}{r!\,(c)_r(c)_{2r}} x^{2r} F(a+r,b+b'+2r;c+2r;x). \quad (17)$$

If in this we put

(i) $c = b+b'$, (ii) $b = \frac{1}{2}c+\frac{1}{2}$, $b' = \frac{1}{2}c-\frac{1}{2}$, (iii) $b = b' = \frac{1}{2}c$, so that $b+b' = c$ throughout, we get respectively

$$F(a,b;b+b';x)F(a,b';b+b';x)$$
$$= (1-x)^{-a}{}_4F_3\left[\begin{matrix} a,b,b',b+b'-a; \\ b+b',\ \frac{1}{2}(b+b'),\frac{1}{2}(b+b'+1)\end{matrix}\ \tfrac{1}{4}x^2/(x-1)\right], \quad (18)$$

$$F(a,\tfrac{1}{2}c+\tfrac{1}{2};c;x)F(a,\tfrac{1}{2}c-\tfrac{1}{2};c;x)$$
$$= (1-x)^{-a}{}_3F_2\left[\begin{matrix} a,c-a,\frac{1}{2}c-\frac{1}{2}; \\ c,\frac{1}{2}c\end{matrix}\ \tfrac{1}{4}x^2/(x-1)\right], \quad (19)$$

$$\{F(a,\tfrac{1}{2}c;c;x)\}^2 = (1-x)^{-a}{}_3F_2\left[\begin{matrix} a,c-a,\frac{1}{2}c; \\ c,\frac{1}{2}c+\frac{1}{2}\end{matrix}\ \tfrac{1}{4}x^2/(x-1)\right]. \quad (20)$$

The formula (18) has been given by Bailey,* who showed† that it could be derived from a result of Watson's‡ which is equivalent to our (**5**) (51).

If in (38), (44), (46) of (**5**) we put $y = x$, using (15) above in the first and third of them, we get

$$F(a,b;c;2x) = \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_r}{r!\,(c)_{2r}} x^{2r} F(a+2r,2b+2r;c+2r;x), \quad (21)$$

$$F(a,b;c;2x-x^2) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{r!\,(c)_r}(-x^2)^r F(a+r,b+r;c+r;2x), \quad (22)$$

$$F(a,b;c;2x-x^2) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c-a)_r}{r!\,(c)_{2r}}(-x^2)^r F(a+r,2b+2r;c+2r;x); \quad (23)$$

and, if in (50) of (**5**) we put $y = -x$, we get

$$F(a,b;c;x^2) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c-a)_r(c-b)_r}{r!\,(c+r-1)_r(c)_{2r}} \times$$
$$\times x^{2r} F(a+r,b+r;c+2r;x)F(a+r,b+r;c+2r;-x). \quad (24)$$

Corresponding inverse formulae can be obtained from the inverse formulae (39), (45), (47), (51) of (**5**).

If in (24) we make a or b equal to c, we get an identity of the form

$$(1-x^2)^{-h} = (1-x)^{-h}(1+x)^{-h}$$

of which (24) can be regarded as a generalization.

* W. N. Bailey (**2**) (6.3). † loc. cit. § 7.

‡ G. N. Watson (**10**) 195 (the last formula).

The expressions (21), (22), (23) are readily obtained if the hypergeometric functions are replaced by the suitable Eulerian integrals. If we make this substitution in (24), we get the equality of integrals

$$\mathrm{B}(a,c-a)\int\limits_0^1 U^{b-1}(1-U)^{c-b-1}(1-Ux^2)^{-a}\,dU$$

$$=\int\limits_0^1\int\limits_0^1 t^{a-1}u^{b-1}\{(1-t)(1+ux)\}^{c-a-1}\{(1-u)(1-tx)\}^{c-b-1}\times$$
$$\times\{(1-tx)(1+ux)-x^2tu(1-t)(1-u)\}^{-c}\times$$
$$\times\{(1-tx)(1+ux)+x^2tu(1-t)(1-u)\}\,dtdu. \quad (25)$$

Replacing the beta function by the Eulerian integral, we get an equality of integrals of the form

$$\int\limits_0^1\int\limits_0^1 A^aB^bC^cD\,dTdU = \int\limits_0^1\int\limits_0^1 \alpha^a\beta^b\gamma^c\delta\,dtdu,$$

where A, B, C, D are functions of T, U (and x) and α, β, γ, δ are functions of t, u (and x). The best-known example of this type comes from the expression of $\mathrm{B}(a,c-a)\mathrm{B}(b,c-b)F(a,b;c;x)$ as a product of Eulerian integrals, i.e. as a double integral, in which interchange of a, b gives a pair of equivalent double integrals. It is always easy to obtain the equivalence by direct transformation of the integrals, for we find that the four relations

$$A=\alpha, \qquad B=\beta, \qquad C=\gamma, \qquad D\frac{\partial(T,U)}{\partial(t,u)}=\delta$$

are consistent: in fact they give a $(1,1)$ transformation between T,U and t,u which transforms the square $0\leqslant T,U\leqslant 1$ into the square $0\leqslant t,u\leqslant 1$. Whether a theory has been worked out of $(1,1)$ transformations that leave the unit square invariant (with extensions to the unit cube, etc.) we do not know.

3. The confluent double hypergeometric functions

If, in the simple hypergeometric function, we write $1/\epsilon$, ϵx for b, x and take the limit $\epsilon\to 0$, we get the confluent hypergeometric function

$$\lim_{\epsilon\to 0}F(a,1/\epsilon;c;\epsilon x) = {}_1F_1(a;c;x).$$

A second limit gives

$$\lim_{\epsilon\to 0}{}_1F_1(1/\epsilon;c;\epsilon x) = {}_0F_1(c;x),$$

where ${}_0F_1$ is expressible as a Bessel's function.

Similar limiting processes in the double hypergeometric functions give, in addition to products of F, ${}_1F_1$, ${}_0F_1$, and functions of $x+y$, the seven confluent double hypergeometric functions Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1, Ξ_2 in Humbert's notation:* in all, fourteen functions, which we can group as follows:

$$\left.\begin{aligned}
\Phi_1(a;b;c;x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{m!\,n!\,(c)_{m+n}}x^m y^n \\
\Psi_1(a;b;c,c';x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{m!\,n!\,(c)_m(c')_n}x^m y^n \\
\Xi_1(a,a';b;c;x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(b)_m(a')_n}{m!\,n!\,(c)_{m+n}}x^m y^n \\
\text{and}\qquad & F(a,b;c;x)\,{}_1F_1(a';c';y);
\end{aligned}\right\}\quad \text{(I)}$$

$$\left.\begin{aligned}
\Phi_2(a,a';c;x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a')_n}{m!\,n!\,(c)_{m+n}}x^m y^n \\
\Psi_2(a;c,c';x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}}{m!\,n!\,(c)_m(c')_n} \\
\text{and}\qquad & {}_1F_1(a;c;x)\,{}_1F_1(a';c';y), \qquad {}_1F_1(a;c;x+y);
\end{aligned}\right\}\quad \text{(II A)}$$

$$\left.\begin{aligned}
\Xi_2(a;b;c;x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(b)_m}{m!\,n!\,(c)_{m+n}}x^m y^n \\
\text{and}\qquad & F(a,b;c;x)\,{}_0F_1(c';y);
\end{aligned}\right\}\quad \text{(II B)}$$

$$\left.\begin{aligned}
\Phi_3(a;c;x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m}{m!\,n!\,(c)_{m+n}}x^m y^n \\
\text{and}\qquad & {}_1F_1(a;c;x)\,{}_0F_1(c';y);
\end{aligned}\right\}\quad \text{(III)}$$

$${}_0F_1(c;x)\,{}_0F_1(c';y) \quad \text{and} \quad {}_0F_1(c;x+y). \qquad \text{(IV)}$$

The grouping is by the suffixes in the numerators: in (I) they are $(m,m;n)$, in (II A) (m,n), in (II B) (m,m), and in (III) merely m; in (IV) the numerators are unity. The numeral indicates the degree of 'confluence', i.e. the number of limits that have been taken. In (II A) and (IV) there is symmetry between the suffixes; the others are 'skew', i.e. one suffix (m) predominates in the numerator. In

* P. Humbert (8), or P. Appell and J. Kampé de Fériet (1) 122–6.

(I), (II B) we need $|x| < 1$ for convergence; the convergence is otherwise unrestricted.

The double confluent functions, like the double functions $F^{(r)}$, can be built up from the elementary functions by use of the symbolic operators ∇, Δ. Thus from (I), (II B), (III) we have for the skew functions

$$\left.\begin{aligned}\Phi_1(a;b;c;x,y) &= \nabla(a)\Delta(c)F(a,b;c;x)\,{}_1F_1(a;c;y)\\ &= \nabla(a)\Xi_1(a,a;b;c;x,y)\\ &= \Delta(c)\Psi_1(a;b;c,c;x,y)\\ \Xi_1(a,a';b;c;x,y) &= \Delta(c)F(a,b;c;x)\,{}_1F_1(a';c;y)\\ \Psi_1(a;b;c,c';x,y) &= \nabla(a)F(a,b;c;x)\,{}_1F_1(a;c';y)\\ \Xi_2(a;b;c;x,y) &= \Delta(c)F(a,b;c;x)\,{}_0F_1(c;y)\\ \Phi_3(a;c;x,y) &= \Delta(c)\,{}_1F_1(a;c;x)\,{}_0F_1(c;y)\end{aligned}\right\};\tag{26}$$

and from (II A), (IV) for the symmetric functions

$$\left.\begin{aligned}{}_1F_1(a;c;x+y) &= \nabla(a)\Delta(c)\,{}_1F_1(a;c;x)\,{}_1F_1(a;c;y)\\ &= \nabla(a)\Phi_2(a,a;c;x,y)\\ &= \Delta(c)\Psi_2(a;c,c;x,y)\\ \Phi_2(a,a';c;x,y) &= \Delta(c)\,{}_1F_1(a;c;x)\,{}_1F_1(a';c;y)\\ \Psi_2(a;c,c';x,y) &= \nabla(a)\,{}_1F_1(a;c;x)\,{}_1F_1(a;c';y)\\ {}_0F_1(c;x+y) &= \Delta(c)\,{}_0F_1(c;x)\,{}_0F_1(c;y)\end{aligned}\right\}.\tag{27}$$

4. Twenty-six expansions

We can obtain expansions of the confluent functions similar to those given in (5) §§ 3, 4 for the functions $F^{(r)}$. These can be derived from the symbolic relations (26), (27) above by using the appropriate expansion for ∇, Δ as described in (5) § 2; or we can take suitable limits in the expansions of (5). In the first argument we can appeal without difficulty to considerations of absolute convergence, since at the worst we need $|x| < 1$, $|y| < \infty$; in the second we note that the passages to the limit are valid since the convergence is uniform as $b \to \infty$, $y \to 0$, etc. In any case the convergence-condition for the resulting expansion is that needed for the existence of the functions involved.

With these remarks we write down the twenty-six expansions without further proof, grouping them as before in inverse pairs, the 'direct' expansion coming first with an *even* reference number. Each

pair is deducible from a single relation of (26), (27), and we can discover which relation to use in each case by noting which functions appear on the two sides of the expansion. On the other hand, in taking limits it may not always be obvious which expansion of (5) should be used, and so to each expansion we prefix in brackets [ ] the number in (5) of the expansion from which it can be derived. For functions in the later groups it is generally possible to obtain the expansion as the limit of an earlier expansion in the series; we then prefix in brackets { } the number of that earlier expansion. In that sense (52), (53) may be regarded as the ultimate limits of all the expansions. The formulae are now arranged with the 'skew' groups (I), (II B), (III) preceding the 'symmetric' groups (II A), (IV).

[26] $\Psi_1(a;b;c,c';x,y)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{r!\,(c)_r(c')_r} x^r y^r F(a+r,b+r;c+r;x)\,{}_1F_1(a+r;c'+r;y), \tag{28}$$

[27] $F(a,b;c;x)\,{}_1F_1(a;c';y)$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r}{r!\,(c)_r(c')_r} x^r y^r \Psi_1(a+r;b+r;c+r;c'+r;x,y), \tag{29}$$

[28] $$\Xi_1(a,a';b;c;x,y) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(a')_r(b)_r}{r!\,(c+r-1)_r(c)_{2r}} \times$$

$$\times x^r y^r F(a+r,b+r;c+2r;x)\,{}_1F_1(a'+r;c+2r;y), \tag{30}$$

[29] $F(a,b;c;x)\,{}_1F_1(a';c;y)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(a')_r(b)_r}{r!\,(c)_r(c)_{2r}} x^r y^r \Xi_1(a+r,a'+r;b+r;c+2r;x,y), \tag{31}$$

[30] $$\Phi_1(a;b;c;x,y) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c-a)_r}{r!\,(c+r-1)_r(c)_{2r}} \times$$

$$\times x^r y^r F(a+r,b+r;c+2r;x)\,{}_1F_1(a+r;c+2r;y), \tag{32}$$

[31] $F(a,b;c;x)\,{}_1F_1(a;c;y)$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(c-a)_r}{r!\,(c)_r(c)_{2r}} x^r y^r \Phi_1(a+r;b+r;c+2r;x,y), \tag{33}$$

[32] $\Phi_1(a;b;c;x,y)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{r!\,(c)_{2r}}x^r y^r \Xi_1(a+r,a+r;b+r;c+2r;x,y), \tag{34}$$

[33] $\Xi_1(a,a;b;c;x,y)$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(b)_r}{r!\,(c)_{2r}}x^r y^r \Phi_1(a+r;b+r;c+2r;x,y), \tag{35}$$

[34] $\Phi_1(a;b;c;x,y)$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_{2r}(b)_r}{r!\,(c+r-1)_r(c)_{2r}}\times$$
$$x^r y^r \Psi_1(a+2r;b+r;c+2r,c+2r;x,y), \tag{36}$$

[35] $\Psi_1(a;b;c,c;x,y)$

$$=\sum_{r=0}^{\infty}\frac{(a)_{2r}(b)_r}{r!\,(c)_r(c)_{2r}}x^r y^r \Phi_1(a+2r;b+r;c+2r;x,y), \tag{37}$$

[28], {30} $\Xi_2(a;b;c;x,y)$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(b)_r}{r!\,(c+r-1)_r(c)_{2r}}\times$$
$$\times x^r y^r F(a+r,b+r;c+2r;x)\,{}_0F_1(c+2r;y), \tag{38}$$

[29], {31} $F(a,b;c;x)\,{}_0F_1(c;y)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{r!\,(c)_r(c)_{2r}}x^r y^r \Xi_2(a+r;b+r;c+2r;x,y), \tag{39}$$

[28], [30], [34], {30}, {38} $\Phi_3(a;c;x,y)$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r}{r!\,(c+r-1)_r(c)_{2r}}\times$$
$$\times x^r y^r\,{}_1F_1(a+r;c+2r;x)\,{}_0F_1(c+2r;y), \tag{40}$$

[29], [31], [35], {31}, {39} ${}_1F_1(a;c;x)\,{}_0F_1(c;y)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{r!\,(c)_r(c)_{2r}}x^r y^r \Phi_3(a+r;c+2r;x,y); \tag{41}$$

[50] ${}_1F_1(a;c;x+y)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(c-a)_r}{r!\,(c+r-1)_r(c)_{2r}}\times$$
$$\times x^r y^r\,{}_1F_1(a+r;c+2r;x)\,{}_1F_1(a+r;c+2r;y), \tag{42}$$

[51] ${}_1F_1(a;c;x)\,{}_1F_1(a;c;y)$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(c-a)_r}{r!\,(c)_r(c)_{2r}}x^ry^r{}_1F_1(a+r;c+2r;x+y), \qquad (43)$$

[38], [48], {34} ${}_1F_1(a;c;x+y)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{r!\,(c)_{2r}}x^ry^r\Phi_2(a+r,a+r;c+2r;x,y), \qquad (44)$$

[39], [49], {35} $\Phi_2(a,a;c;x,y)$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r}{r!\,(c)_{2r}}x^ry^r{}_1F_1(a+r;c+2r;x+y), \qquad (45)$$

[40], [42], {36} ${}_1F_1(a;c;x+y)$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_{2r}}{r!\,(c+r-1)_r(c)_{2r}}x^ry^r\Psi_2(a+2r;c+2r,c+2r;x,y), \qquad (46)$$

[41], [43], {37} $\Psi_2(a;c,c;x,y)$

$$=\sum_{r=0}^{\infty}\frac{(a)_{2r}}{r!\,(c)_r(c)_{2r}}x^ry^r{}_1F_1(a+2r;c+2r;x+y), \qquad (47)$$

[28], [30], {30} $\Phi_2(a,a';c;x,y)$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(a')_r}{r!\,(c+r-1)_r(c)_{2r}}\times$$
$$\times x^ry^r{}_1F_1(a+r;c+2r;x)\,{}_1F_1(a'+r;c+2r;y), \qquad (48)$$

[29], [31], {31} ${}_1F_1(a;c;x)\,{}_1F_1(a';c;y)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(a')_r}{r!\,(c)_r(c)_{2r}}x^ry^r\Phi_2(a+r,a'+r;c+2r;x,y), \qquad (49)$$

[26], [54], {28} $\Psi_2(a;c,c';x,y)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{r!\,(c)_r(c')_r}x^ry^r{}_1F_1(a+r;c+r;x)\,{}_1F_1(a+r;c'+r;y), \qquad (50)$$

[27], [55], {29} ${}_1F_1(a;c;x)\,{}_1F_1(a;c';y)$

$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r}{r!\,(c)_r(c')_r}x^ry^r\Psi_2(a+r;c+r;c'+r;x,y), \qquad (51)$$

[28], [30], [34], [40], [42], {30}, {32}, {36}, {38}, {40}, {42}, {46}, {48}

$$ {}_0F_1(c;x+y) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(c+r-1)_r(c)_{2r}} x^r y^r {}_0F_1(c+2r;x)\,{}_0F_1(c+2r;y), \tag{52} $$

[29], [31], [35], [41], [43], {31}, {33}, {37}, {39}, {41}, {43}, {47}, {49}

$$ {}_0F_1(c;x)\,{}_0F_1(c;y) = \sum_{r=0}^{\infty} \frac{1}{r!\,(c)_r(c)_{2r}} x^r y^r {}_0F_1(c+2r;x+y). \tag{53} $$

The expansion (43) above has been given by Bailey,* who deduces it as the limit of the expansion, equivalent to our (**5**) (51), given by Watson,† which we have already quoted.

We remark the absence of any formula connecting Ξ_1 with Ψ_1, or Φ_2 with Ψ_2. This corresponds to the absence, in (**5**), of any formula connecting $F^{(2)}$ with $F^{(3)}$.

5. Definite integrals

If we substitute Euler's integral

$$ F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b}\,dt \tag{54} $$

and its limiting form

$$ {}_1F_1(a;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tx}\,dt \tag{55} $$

for F and ${}_1F_1$ where they occur on the right of the foregoing expansions, we obtain definite integrals for the corresponding confluent hypergeometric functions. The only integral with an elementary integrand comes out as

$$ \Psi_1(a;b;c,c';x,y) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-b)\Gamma(c'-a)} \times $$
$$ \times \int_0^1\int_0^1 u^{b-1}v^{a-1}(1-u)^{c-b-1}(1-v)^{c'-a-1}(1-ux)^{-a}\exp\left(\frac{vy}{1-ux}\right) du dv. \tag{56} $$

* W. N. Bailey (**3**), 218 (5.1).

† G. N. Watson (**10**) 195 (the last formula).

Humbert has given* definite integrals for the functions Φ_1, Φ_2, Ξ_1 which may be regarded as limiting forms of the known† integrals for $F^{(1)}$, $F^{(3)}$. In the same way the above integral for Ψ_1, which does not appear to have been given by Humbert, can be obtained, as the limiting form, when $b' \to \infty$,‡ of the integral

$$F^{(2)}[a;b,b';c,c';x,y] = \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-b)\Gamma(c'-a)} \times$$
$$\times \int_0^1\int_0^1 u^{b-1}v^{a-1}(1-u)^{c-b-1}(1-v)^{c'-a-1}(1-ux)^{b'-a}(1-ux-vy)^{-b'}\,dudv. \tag{57}$$

This integral (57), which is an alternative to that usually given§ for $F^{(2)}$, can be obtained from (**5**) (26) by use of (54) with suitable choice of parameters.

There do not appear to be similar 'elementary' integrals for Ψ_2, Ξ_2, Φ_3, but it is possibly worth while to record the following integrals involving the Bessel's function ${}_0F_1$, obtainable respectively from (50), (47), (45), (43),

$$\Psi_2(a;c,c';x,y) = \frac{\Gamma(c)\Gamma(c')}{\{\Gamma(a)\}^2\Gamma(c-a)\Gamma(c'-a)} \times$$
$$\times \int_0^1\int_0^1 (uv)^{a-1}(1-u)^{c-a-1}(1-v)^{c'-a-1}e^{(u+v)x}{}_0F_1(a;xyuv)\,dudv, \tag{58}$$

$$\Psi_2(a;c,c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}e^{ux}{}_0F_1(c;xyu^2)\,du, \tag{59}$$

$$\Phi_2(a,a;c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \times$$
$$\times \int_0^1 u^{a-1}(1-u)^{c-a-1}e^{u(x+y)}{}_0F_1[c-a;-xyu(1-u)]\,du, \tag{60}$$

$${}_1F_1(a;c;x)\,{}_1F_1(a;c;y)$$
$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}e^{u(x+y)}{}_0F_1[c;-xyu(1-u)]\,du. \tag{61}$$

* P. Humbert (**8**), 79 (chapter IV); the integrals for Φ_1, Φ_2 are quoted in P. Appell and J. Kampé de Fériet (**1**), 127 (18), 128 (19).

† See, for example, W. N. Bailey (**4**), 76–7, § 9.3, or P. Appell and J. Kampé de Fériet (**1**), 28–33, chap. 2.

‡ We understand, of course, that y has been replaced by y/b' before the limit is taken; and so always.

§ W. N. Bailey (**4**), 77 (2), or P. Appell and J. Kampé de Fériet (**1**), 28 (2).

6. Some further formulae

6.1. The limiting form, as $b \to \infty$, of the identity

$$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x)$$

is
$$ {}_1F_1(a;c;x) = e^x {}_1F_1(c-a;c;-x), \tag{62}$$

where the change of sign in the argument of x will be noted. If we compare (30), (32) using this identity (62), we deduce*

$$\Phi_1(a;b;c;x,y) = e^y \Xi_1(a,c-a;b;c;x,-y). \tag{63}$$

Similar comparison of (42), (48) gives

$${}_1F_1(a;c;x+y) = e^y \Phi_2(a,c-a;c;x,-y), \tag{64}$$

which is the limiting form of (63) as $b \to \infty$.

6.2. Writing $y = x$ in the definition of Φ_2 and using Vandermonde's lemma

$$\sum_{m+n=N} \frac{(a)_m(a')_n}{m!\,n!} = \frac{(a+a')_N}{N!},$$

we have
$$\Phi_2(a,a';c;x,x) = {}_1F_1(a+a';c;x). \tag{65}$$

Similarly, the lemma

$$\sum_{m+n=N} \frac{1}{m!\,n!\,(c)_m(c')_n} = \frac{2^{2N}(\frac{1}{2}c+\frac{1}{2}c')_N(\frac{1}{2}c+\frac{1}{2}c'-\frac{1}{2})_N}{N!\,(c)_N(c')_N(c+c'-1)_N}$$

gives
$$\Psi_2(a;c,c';x,x) = {}_3F_3\left[\begin{matrix} a, & \frac{1}{2}c+\frac{1}{2}c', & \frac{1}{2}c+\frac{1}{2}c'-\frac{1}{2}; \\ c, & c', & c+c'-1 \end{matrix}\; 4x\right], \tag{66}$$

and, in particular,

$$\Psi_2(a;a,a;x,x) = {}_1F_1(a-\tfrac{1}{2};2a-1;4x), \tag{67}$$

$$\left.\begin{matrix}\Psi_2(2c-2;c,c-1;x,x) \\ \Psi_2(2c-1;c,c;x,x)\end{matrix}\right\} = {}_1F_1(c-\tfrac{1}{2};c;4x). \tag{68}$$

Using (65) in (44), (45), (48), (49) we get

$${}_1F_1(a;c;2x) = \sum_{r=0}^{\infty} \frac{(a)_r}{r!\,(c)_{2r}} x^{2r} {}_1F_1(2a+2r;c+2r;x), \tag{69}$$

$${}_1F_1(2a;c;x) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r}{r!\,(c)_{2r}} x^{2r} {}_1F_1(a+r;c+2r;2x), \tag{70}$$

* Given by P. Humbert (**8**), 77 (the final formula of chapter II), with an incorrect sign; quoted correctly by P. Appell and J. Kampé de Fériet (**1**), 127 (17).

$$ {}_1F_1(a+a';c;x) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(a')_r}{r!\,(c+r-1)_r(c)_{2r}} x^{2r}{}_1F_1(a+r;c+2r;x)\times $$
$$ \times {}_1F_1(a'+r;c+2r;x), \quad (71) $$

$$ {}_1F_1(a;c;x)\,{}_1F_1(a';c;x) = \sum_{r=0}^{\infty} \frac{(a)_r(a')_r}{r!\,(c)_r(c)_{2r}} x^{2r}{}_1F_1(a+a'+2r;c+2r;x). \quad (72) $$

Writing $c = 2a$ in (69), (70) and noting that

$$ (a)_r x^{2r}/(2a)_{2r} = (\tfrac{1}{2}x)^{2r}/(a+\tfrac{1}{2})_r, $$

we have

$$ {}_1F_1(a;2a;2x) = e^x{}_0F_1(a+\tfrac{1}{2};\tfrac{1}{4}x^2), \quad (73) $$

$$ e^x = \sum_{r=0}^{\infty} (-)^r \frac{(\frac{1}{2}x)^{2r}}{r!\,(a+\frac{1}{2})_r}\,{}_1F_1(a+r;2a+2r;2x), \quad (74) $$

of which the former is quoted by Watson* as due to Kummer. Similarly, if we write $c = 2a = 2a'$ in (72), we get

$$ \{{}_1F_1(a;2a;x)\}^2 = e^x{}_1F_2\begin{bmatrix} a; \\ a+\frac{1}{2}, 2a \end{bmatrix} \tfrac{1}{4}x^2. \quad (75) $$

If we substitute for $e^{-x}{}_1F_1$ from (73) in (74), we have

$$ 1 = \sum_{r=0}^{\infty} (-)^r \frac{(\frac{1}{2}x)^{2x}}{r!\,(a+\frac{1}{2})_r}\,{}_0F_1(a+r+\tfrac{1}{2};\tfrac{1}{4}x^2). \quad (76) $$

Remembering that

$$ {}_0F_1(c;\tfrac{1}{4}x^2) = \Gamma(c)(\tfrac{1}{2}x)^{1-c}I_{c-1}(x), \quad (77) $$

we write (76) as

$$ (\tfrac{1}{2}x)^\nu = \Gamma(\nu+1) \sum_{r=0}^{\infty} (-)^r \frac{(\frac{1}{2}x)^r}{r!} I_{\nu+r}. \quad (78) $$

If in (52), (53) we replace x, y, $x+y$, c by $\frac{1}{4}x^2$, $\frac{1}{4}y^2$, $\frac{1}{4}R^2$, $\nu+1$ and use (77), we get the inverse pair of formulae

$$ I_\nu(R) = \left(\frac{2R}{xy}\right)^\nu \sum_{r=0}^{\infty} (-)^r \frac{(\nu+2r)\Gamma(\nu+r)}{r!} I_{\nu+2r}(x)I_{\nu+2r}(y), \quad (79) $$

$$ I_\nu(x)I_\nu(y) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}xy/R)^{\nu+2r}}{r!\,\Gamma(\nu+r+1)} I_{\nu+2r}(R), \quad (80) $$

where $R^2 \equiv x^2+y^2$. The first of these is covered by Gegenbauer's addition theorem,† and the second is given by Bailey.‡

* G. N. Watson (**9**), 101 (1) or (2).

† See, for instance, G. N. Watson (**9**), 363 (2) or 366 (13).

‡ W. N. Bailey (**3**), 219 (5.2).

6.3. If we similarly use (67) or (68) in (46), (47), (50), (51), we get results that appear less interesting. Thus (67) in (46) with (73) gives the alternative to (76)

$$1 = \sum_{r=0}^{\infty} (-)^r \frac{(\frac{1}{2}x)^{2r}}{r!\,(a+r-1)_r} {}_0F_1(a+2r; \tfrac{1}{4}x^2), \tag{81}$$

i.e.

$$(\tfrac{1}{2}x)^{\nu} = \sum_{r=0}^{\infty} (-)^r \frac{(\nu+2r)\Gamma(\nu+r)}{r!} I_{\nu+2r}(x), \tag{82}$$

a result also due to Gegenbauer.* Again (67) in (47) or (50) reproduces (73) as a particular case of the formula

$$\Psi_2(c; c, c; x, y) = e^{x+y} {}_0F_1(c; xy). \tag{83}$$

Finally (68_2) in (47), (50) and (68_1) in (50) give three expansions for ${}_1F_1(c-\frac{1}{2}; c; 4x)$, namely

$${}_1F_1(c-\tfrac{1}{2}; c; 4x) = \sum_{r=0}^{\infty} \frac{(2c-1)_{2r}}{r!\,(c)_r(c)_{2r}} x^{2r} {}_1F_1(2c+2r-1; c+2r; 2x) \tag{84}$$

$$= \sum_{r=0}^{\infty} \frac{(2c-1)_r}{r!\,\{(c)_r\}^2} x^{2r} \{{}_1F_1(2c+r-1; c+r; x)\}^2 \tag{85}$$

$$= \sum_{r=0}^{\infty} \frac{(2c-2)_r}{r!\,(c)_r(c-1)_r} x^{2r} {}_1F_1(2c+r-2; c+r; x) \times$$

$$\times {}_1F_1(2c+r-2; c+r-1; x). \tag{86}$$

7. Whittaker's functions and Laguerre polynomials

Whittaker's function $M_{k,\mu}$ and Laguerre's generalized polynomial L_n^{α} being defined as

$$M_{k,\mu}(x) = x^{\mu+\frac{1}{2}} \exp(-\tfrac{1}{2}x) {}_1F_1(\mu-k+\tfrac{1}{2}; 2\mu+1; x), \tag{87}$$

$$L_n^{\alpha}(x) = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)} {}_1F_1(-n; \alpha+1; x), \tag{88}$$

it is clear that any formula involving ${}_1F_1$ can be restated in terms of either of these functions (with the restriction in the case of

* Cf. G. N. Watson (9), 138 (1).

Laguerre's polynomial that n is an integer). Thus (42), (43) give the inverse pairs

$$M_{k,\mu}(x+y) = \left(\frac{x+y}{xy}\right)^{\mu+\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(\mu-k+\frac{1}{2})_r(\mu+k+\frac{1}{2})_r}{r!\,(2\mu+r)_r(2\mu+1)_{2r}} M_{k,\mu+r}(x)M_{k,\mu+r}(y), \tag{89}$$

$$M_{k,\mu}(x)M_{k,\mu}(y) = \left(\frac{xy}{x+y}\right)^{\mu+\frac{1}{2}} \sum_{r=0}^{\infty} (-)^r \frac{(\mu-k+\frac{1}{2})_r(\mu+k+\frac{1}{2})_r}{r!\,(2\mu+1)_r(2\mu+1)_{2r}} \times \\ \times \left(\frac{xy}{x+y}\right)^r M_{k,\mu+r}(x+y), \tag{90}$$

$$L_n^\alpha(x+y) = \sum_{r=0}^{n} (-)^r \frac{(n-r)!\,(\alpha+2r)\Gamma(\alpha+r)}{r!\,\Gamma(\alpha+n+r+1)} x^r y^r L_{n-r}^{\alpha+2r}(x)L_{n-r}^{\alpha+2r}(y), \tag{91}$$

$$L_n^\alpha(x)L_n^\alpha(y) = \frac{\Gamma(\alpha+n+1)}{n!} \sum_{r=0}^{n} \frac{x^r y^r}{r!\,\Gamma(\alpha+r+1)} L_{n-r}^{\alpha+2r}(x+y). \tag{92}$$

The last of these has been given by Bailey.*

For the L-functions, (69), (70) give

$$L_n^\alpha(2x) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+2n+1)} \sum_{r=0}^{n} (-)^r \frac{(2n-2r)!}{r!\,(n-r)!} x^{2r} L_{2(n-r)}^{\alpha+2r}(x), \tag{93}$$

$$L_{2n}^\alpha(x) = \frac{n!\,\Gamma(\alpha+2n+1)}{(2n)!} \sum_{r=0}^{n} \frac{x^{2r}}{r!\,\Gamma(\alpha+n+r+1)} L_{n-r}^{\alpha+2r}(2x); \tag{94}$$

the corresponding results for the M-functions appear less interesting. Again (71), (72) give

$$M_{K,\mu}(x) = x^{-(\mu+\frac{1}{2})}\exp(\tfrac{1}{2}x) \times \\ \times \sum_{r=0}^{\infty} (-)^r \frac{(\mu-k+\frac{1}{2})_r(\mu-k'+\frac{1}{2})_r}{r!\,(2\mu+r)_r(2\mu+1)_{2r}} M_{k,\mu+r}(x)M_{k',\mu+r}(x), \tag{95}$$

$$M_{k,\mu}(x)M_{k',\mu}(x) = x^{\mu+\frac{1}{2}}\exp(-\tfrac{1}{2}x) \times \\ \times \sum_{r=0}^{\infty} \frac{(\mu-k+\frac{1}{2})_r(\mu-k'+\frac{1}{2})_r}{r!\,(2\mu+1)_r(2\mu+1)_{2r}} x^r M_{K-r,\mu+r}(x), \tag{96}$$

where $K \equiv k+k'-\mu-\frac{1}{2}$; and

$$L_{m+n}^\alpha(x) = \frac{m!\,n!\,\Gamma(\alpha+m+n+1)}{(m+n)!} \times \\ \times \sum_{r=0}^{\min(m,n)} (-)^r \frac{(\alpha+2r)\Gamma(\alpha+r)x^{2r}}{\Gamma(\alpha+m+r+1)\Gamma(\alpha+n+r+1)} L_{m-r}^{\alpha+2r}(x)L_{n-r}^{\alpha+2r}(x), \tag{97}$$

* W. N. Bailey (3), 219 (5.4).

$$L_m^\alpha(x)L_n^\alpha(x) = \frac{\Gamma(\alpha+m+1)\Gamma(\alpha+n+1)}{\Gamma(\alpha+m+n+1)}\times$$
$$\times\sum_{r=0}^{\min(m,n)}\frac{(m+n-2r)!\,x^{2r}}{r!\,(m-r)!\,(n-r)!\,\Gamma(\alpha+r+1)}L_{m+n-2r}^{\alpha+2r}(x). \quad (98)$$

Of these (96) and (98) have been given by Erdélyi* and (98) by Howell.†

The formulae (73)–(75) give results for $M_{0,\mu}$, of which we may record

$$x^{\mu+\frac{1}{2}} = \sum_{r=0}^{\infty}\frac{(-\frac{1}{4}x)^r}{r!\,(\mu+1)_r}M_{0,\mu+r}(2x), \quad (99)$$

while (84)–(86) give alternative expansions for $M_{-\mu,\mu}$. The corresponding formulae in the L-function seem more tedious.

* A. Erdélyi (**6**), 144 (4, 3) and 145 (4, 5).
† W. T. Howell (**7**), 402 (29) with a misprint.

REFERENCES

1. P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques* (Paris, 1926).
2. W. N. Bailey, 'Some theorems concerning products of hypergeometric series': *Proc. London Math. Soc.* (2), 38 (1935), 377–84.
3. —— 'On the product of two Legendre polynomials with different arguments': *Proc. London Math. Soc.* (2), 41 (1936), 215–20.
4. —— *Generalized Hypergeometric Series* (Cambridge, 1935).
5. J. L. Burchnall and T. W. Chaundy, 'Expansions of Appell's double hypergeometric functions': *Quart. J. of Math.* (Oxford), 11 (1940), 249–70.
6. A. Erdélyi, 'Untersuchungen über Produkte von Whittakerschen Funktionen': *Monats. für Math. und Phys.* 46 (1937–8), 132–56.
7. W. T. Howell, 'On products of Laguerre polynomials': *Phil. Mag.* (7), 24 (1937), 396–405.
8. P. Humbert, 'The confluent hypergeometric functions of two variables': *Proc. Roy. Soc. Edinburgh*, 41 (1920–1), 73–96.
9. G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1922).
10. —— 'The product of two hypergeometric functions': *Proc. London Math. Soc.* (2), 20 (1922), 189–95.